Soft-Collinear Effective Theory

Part I: The strategy of regions

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Literature

Unfortunately, no SCET review article is available and to make matters worse two different formalisms are commonly used. A few selected original references are:

- Original papers (using the label formalism):
  - C.W. Bauer, S. Fleming, D. Pirjol and I.W. Stewart, *An effective field theory for collinear and soft gluons: Heavy to light decays,* PRD 63, 114020 (2001) [hep-ph/0011336]
  - C.W. Bauer, S. Fleming, D. Pirjol and I.W. Stewart, *Soft-Collinear Factorization in Effective Field Theory*, PRD 65, 054022 (2002) [hep-ph/0109045]
- SCET in position space:
  - M. Beneke, A.P. Chapovsky, M. Diehl and T. Feldmann,``Soft-collinear effective theory and heavy-to-light currents beyond leading power," Nucl. Phys. B643, 431 (2002) [arXiv:hep-ph/0206152]

Literature

Collider physics applications:

- Factorization analysis for DIS, Drell-Yan and other processes
  - C.W. Bauer, S. Fleming, D. Pirjol, I.Z. Rothstein and I.W. Stewart, *Hard scattering factorization from effective field theory*, PRD 66, 014017 (2002) [hep-ph/0202088].
- Threshold resummation in momentum space using RG evolution
  - TB and M. Neubert, *Threshold resummation in momentum space from effective field theory*, PRL 97, 082001 (2006), [hep-ph/0605050]
- EFT analysis of the IR structure of gauge theory amplitudes
  - TB and M. Neubert, *Infrared singularities of scattering amplitudes in perturbative QCD*, PRL 102, 162001 (2009) [0901.0722]; *On the Structure of Infrared Singularities of Gauge-Theory Amplitudes*, JHEP 0906, 081 (2009) [0903.1126]; *Infrared singularities of QCD amplitudes with massive partons*, PRD 79, 125004 (2009) [0904.1021]

Outline

Part I: The strategy of regions

Part II: Scalar SCET

Part III: Generalization to QCD

Part IV: Resummation by RG evolution

Part V: IR divergences of gauge theory amplitudes

Introduction

IR singularities*

On-shell amplitudes in theories with massless particles (such as gauge theories) have infrared (IR) divergences. They arise from configurations of soft and collinear loop momenta.

In physical observables, they cancel against real radiation contributions involving soft and collinear emissions.

To understand the structure of these singularities in QCD, we thus need to understand the theory in the soft and collinear limit.

* Here: IR refers to soft or collinear. Lorenzo Magnea reserves IR for soft divergences.

Soft-Collinear Effective Theory (SCET)

As the name suggests, this effective field theory (EFT) is constructed to reproduce QCD in the limit where particles become soft or collinear.

Each QCD field is represented by several fields in SCET:

- Soft field
- Collinear fields describe partons propagating with large energy along some reference directions. (*n* reference directions for an *n*-jet process)

Diagrammatic vs. EFT approach

Instead of using an effective field theory, we can just expand the full theory diagrams around the given limit. →Lorenzo Magnea's lecture

There is a close relation between the expanded full theory diagrams and the effective theory

- Soft and collinear regions are reproduced by the EFT diagrams
- High energy region is abosorbed into Wilson coefficients (i.e. coupling constants) of the EFT

Strategy of regions

Beneke and Smirnov NPB522, 321 (1998) [hep-ph/9711391]

- A technique to perform asymptotic expansions of loop integrals in dimensional regularization around various limit.
- The expansion is obtained by splitting the momentum integration into different regions.
- There is a one-to-one correspondence between the strategy of regions technique and Feynman diagrams of EFTs in dimensional regularization

Advantages of the EFT approach

Convenient language to derive all order results

- Factorization theorems
- Renormalization group: resummation of logarithmically enhanced contributions

Gauge invariance is manifest on the Lagrangian level and it is possible to go to higher twist by systematically including subleading terms in the effective Lagrangian.

Strategy of regions

A simple example

Consider the integral

$$I = \int_0^\infty dk \, \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{\ln \frac{M}{m}}{M^2 - m^2}$$

$$= \frac{\ln \frac{M}{m}}{M^2} \left\{ 1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \ldots \right\} \quad \text{for} \quad m^2 \ll M^2$$

How can we expand the integral *before* performing the integration?

Naive expansion

The naive expansion of the integrand leads to ill-defined expressions:

$$I = \int_0^\infty dk \, \frac{k}{(k^2+m^2)(k^2+M^2)}$$

$$\neq \int_0^\infty dk \, \frac{k}{k^2(k^2+M^2)} \left\{ 1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \dots \right\}$$

IR divergence! For k ~ m, the expansion was not justified.

To be expected, since the result depends non-analytically on m/M...

Regions

Split integration into two regions $m \ll \Lambda \ll M$

$$I = \left[\int_0^\Lambda dk + \int_\Lambda^\infty dk \right] \frac{k}{(k^2+m^2)(k^2+M^2)}$$

(I) + (II)

In the low-energy region (I), $k \sim m \ll M$, expand:

$$I_{(\mathrm{I})} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)M^2} \left\{ 1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \dots \right\}$$

Here, Λ acts as a UV cut-off.

Regions

In the high-energy region (II), $m \ll k \sim M$, expand:

$$I_{(\mathrm{II})} = \int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_\Lambda^\infty dk \frac{k}{k^2(k^2+M^2)} \left\{ 1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \dots \right\}$$

Λ acts as a IR cut-off.

Result:

$$\left.\begin{aligned} I_{(\mathrm{I})} &= -\frac{\Lambda^2}{2M^4} - \frac{1}{M^2} \ln \frac{m}{\Lambda}, \\ I_{(\mathrm{II})} &= +\frac{\Lambda^2}{2M^4} - \frac{1}{M^2} \ln \frac{\Lambda}{M}, \end{aligned}\right\} \quad I = I_{(\mathrm{I})} + I_{(\mathrm{II})} = -\frac{1}{M^2} \ln \frac{m}{M} \quad \checkmark$$

The dependence on the regulator (or better separator) Λ has cancelled among the different regions, and we obtain the correct expansion of the integral.

While the method works, putting hard cut-offs between the momentum regions is impractical for complicated loop integrals.

Fortunately, one can get the same result using dimensional regularization!

Dimensional regularization

Consider our integral in "dimensional regularization"

$$I = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)(k^2+M^2)}$$

and calculate the contributions obtained by expanding the integral in regions (I) and (II), but without putting an explicit cutoff Λ.

Dimensional regulator ε regulates the IR and UV divergences which appear in the expanded integrals.

Result:

$$I_{(\mathrm{I})} = \int_0^\infty dk\, k^{-\varepsilon}\, \frac{k}{(k^2+m^2)M^2} \left\{ 1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \dots \right\}$$

$$= \frac{m^{-\epsilon}}{M^2} \left\{ \frac{1}{\epsilon} + \mathcal{O}(\epsilon) \right\} + \dots = \frac{1}{M^2} \left\{ \frac{1}{\epsilon} - \ln m + \mathcal{O}(\epsilon) \right\} + \dots$$

UV div.

$$I_{(\mathrm{II})} = \int_0^\infty dk\, k^{-\varepsilon}\, \frac{k}{k^2(k^2+M^2)} \left\{ 1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \dots \right\}$$

$$= \frac{M^{-\epsilon}}{M^2} \left\{ -\frac{1}{\epsilon} + \mathcal{O}(\epsilon) \right\} + \dots = \frac{1}{M^2} \left\{ -\frac{1}{\epsilon} + \ln M \right\} + \dots$$

IR div.

$$I = I_{(\mathrm{I})} + I_{(\mathrm{II})} = -\frac{1}{M^2} \ln \frac{m}{M} + \dots$$

✓

At first sight, it is surprising that the procedure works. It looks like we are double counting, since we integrate both contributions over the full phase space!

However, note that the two parts scale differently. The low energy integrals behave as $m^{-\varepsilon}$, the high-energy integrals as $M^{-\varepsilon}$.

Full integral for arbitrary ε

(I) + (II)

$$I = \Gamma\left(1 - \frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right)\frac{m^{-\varepsilon} - M^{-\varepsilon}}{M^2 - m^2}$$

Keeping an explicit cutoff would generate additional $\Lambda^{-\varepsilon}$ pieces, which have to cancel among the two parts since the full integral is Λ independent. E.g.

$$I_{(\mathrm{I})} = \int_0^\Lambda dk\, k^{-\varepsilon}\, \frac{k}{(k^2+m^2)M^2}\left\{1 - \frac{k^2}{M^2} + \dots\right\}$$

$$= \left[\int_0^\infty dk - \int_\Lambda^\infty dk\right] k^{-\varepsilon}\, \frac{k}{(k^2+m^2)M^2}\left\{1 - \frac{k^2}{M^2} + \dots\right\}$$

Cut-off part:

$$\int_\Lambda^\infty dk\, k^{-\varepsilon}\, \frac{k}{(k^2+m^2)M^2}\left\{1 - \frac{k^2}{M^2} + \dots\right\} = \int_\Lambda^\infty dk\, k^{-\varepsilon}\, \frac{k}{k^2\, M^2}\left\{1 - \frac{m^2}{k^2} - \frac{k^2}{M^2} + \dots\right\} \sim \Lambda^{-\varepsilon}$$

Since the $\Lambda^{-\varepsilon}$ pieces cancel in the end, we might as well leave them out from the beginning...

Strategy of regions

For a review review: V.A. Smirnov Springer, Tracts Mod.Phys.177:1-262,2002

In general, the expansion is obtained as follows:

- Identify all regions of the integration which lead to singularities in the limit under consideration.
- Expand the integrand in each region and integrate over the *full* phase space.
- Summing the contribution from the different regions gives the expansion of the original integral.

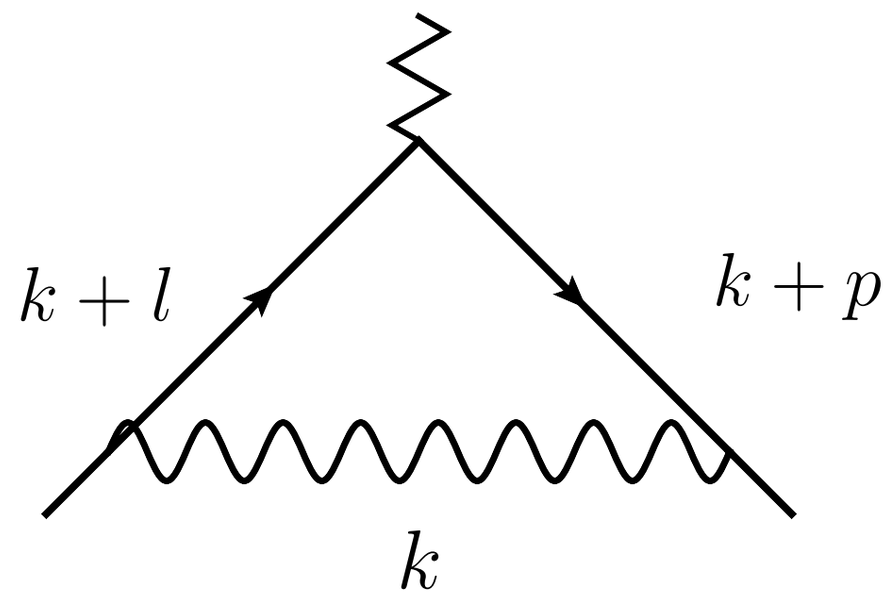
"Problems in the strategy of regions"

V.A. Smirnov, Phys. Lett. B465:226-234,1999

- Need to identify all regions of the integration which lead to singularities.
  - There are examples where additional regions appear at higher loop order.
- Make sure the expanded integrals are regularized.
  - Sometimes dimensional regularization is not enough. Can introduce additional analytic regulators or perform subtractions.
- Avoid double counting...

Application to the Sudakov problem

Let us now perform the expansion in a situation, where particles have large energies, but small invariant masses. Simplest example is the integral



$$L^2 \equiv -l^2 - i0\,, \qquad P^2 \equiv -p^2 - i0\,, \qquad Q^2 \equiv -(l-p)^2 - i0$$

We consider the limit $L^2 \sim P^2 \ll Q^2$.

$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,[(k+l)^2+i0]\,[(k+p)^2+i0]}$$

We consider the scalar integral *I*, but the same momentum regions appear in tensor integrals.

To obtain the expansion introduce light-like reference vectors in the directions of p and l

$$n_\mu = (1,0,0,1) \qquad \bar{n}_\mu = (1,0,0,-1)$$

$$n^2 = \bar{n}^2 = 0 \qquad n\cdot\bar{n} = 2$$

Any vector can be decomposed as

$$p^\mu = (n\cdot p)\,\frac{\bar{n}^\mu}{2} + (\bar{n}\cdot p)\,\frac{n^\mu}{2} + p_\perp^\mu \equiv p_+^\mu + p_-^\mu + p_\perp^\mu\,,$$

Introduce expansion parameter $\lambda^2 \sim P^2/Q^2 \sim L^2/Q^2$

The different components of p^μ scale differently.
Since

$$p^2 = n \cdot p \; \bar{n} \cdot p + p_\perp^2 \sim \lambda^2 Q^2$$

and $p^\mu \approx \frac{1}{2} Q \, n^\mu$, we must have

$$(n \cdot p \, , \, \bar{n} \cdot p \, , \, p_\perp)$$

$$p^\mu \sim \left(\lambda^2 \; , \; 1 \; , \; \lambda\right) Q$$

$$l^\mu \sim \left(1 \; , \; \lambda^2 \; , \; \lambda\right) Q$$

Regions in the Sudakov problem

The following momentum regions contribute to the expansion of the integral

$$(n \cdot k\,,\, \bar{n} \cdot k\,,\, k_\perp)$$

- Hard (h) $\quad k^\mu \sim \left(1 \;\;,\; 1 \;\;,\; 1 \;\right) Q$
- Collinear to p (c1) $\quad k^\mu \sim \left(\lambda^2 \;,\; 1 \;\;,\; \lambda \;\right) Q$
- Collinear to l (c2) $\quad k^\mu \sim \left(1 \;\;,\; \lambda^2 \;,\; \lambda \;\right) Q$
- Soft (s) $\quad k^\mu \sim \left(\lambda^2 \;,\; \lambda^2 \;,\; \lambda^2\right) Q$

All other possible scalings (λ^a, λ^b, λ^c) lead to scaleless integrals upon expanding.

Soft region

Note that the soft region has

$$p_s^2 \sim \lambda^4 Q^2 \sim \frac{L^2 P^2}{Q^2}$$

Interestingly, loop diagrams involve a lower scale than what is present on the external lines. Sometimes scaling $p_s^2 \sim \lambda^4 Q^2$ is called *ultra*-soft.

Implies e.g. that jet-production processes can involve non-perturbative physics, even when the masses of the jets are perturbative.

Low energy regions

In contrast to expansion problems in Euclidean space, we encounter several low-energy regions. Each one is represented by a field in SCET.

Q	QCD	hard	H
M	SCET	collinear	J
M^2/Q	SET	soft	S

Hard contribution

$$
\begin{aligned}
I_{\rm h} &= i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,(k^2+2k_-\cdot l_+ + i0)\,(k^2+2k_+\cdot p_- + i0)} \\
&= \frac{\Gamma(1+\epsilon)}{2l_+\cdot p_-}\,\frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{\mu^2}{2l_+\cdot p_-}\right)^{\epsilon} \\
&= \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}\right)+O(\epsilon)\,,
\end{aligned}
$$

IR divergences!

- Have expanded away small momentum components

$$
p^\mu \to (\bar{n}\cdot p)\,\frac{n^\mu}{2} \equiv p_-^\mu\,, \qquad\qquad l^\mu \to (n\cdot l)\,\frac{\bar{n}^\mu}{2} \equiv l_+^\mu
$$

- The hard region is given by the on-shell form factor integral.

Collinear contribution

$$
\begin{aligned}
I_{\mathrm{c1}} &= i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,(2k_-\cdot l_+ + i0)\,[(k+p)^2+i0]} \\
&= -\frac{\Gamma(1+\epsilon)}{2l_+\cdot p_-}\,\frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{\mu^2}{P^2}\right)^{\epsilon} \\
&= \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}\right)+O(\epsilon)\,.
\end{aligned}
$$

- The other collinear contribution I_{c2} is obtained from exchanging $l \leftrightarrow p$.
- Have expanded $(k+l)^2 = 2k_-\cdot l_+ + \mathcal{O}(\lambda^2)$
- Scales as $(P^2)^{-\varepsilon}$

Soft contribution

$$
\begin{aligned}
I_{\rm s} &= i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,(2k_-\cdot l_+ + l^2 + i0)\,(2k_+\cdot p_- + p^2 + i0)} \\
&= -\frac{\Gamma(1+\epsilon)}{2l_+\cdot p_-}\,\Gamma(\epsilon)\,\Gamma(-\epsilon)\,\left(\frac{2\mu^2 l_+\cdot p_-}{L^2P^2}\right)^{\epsilon} \\
&= \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2Q^2}{L^2P^2}+\frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2}+\frac{\pi^2}{6}\right)+O(\epsilon)\,.
\end{aligned}
$$

UV divergences!

- Scales as $(\Lambda_{\rm soft}{}^2)^{-\varepsilon} \sim (P^2\,L^2/Q^2)^{-\varepsilon}$.
- Expand $(k+p)^2 = 2k_+\cdot p_- + p^2 + \mathcal{O}(\lambda^3)$

Grand total

$$I_{\rm h} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6}\right)$$

$$I_{\rm c1} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P^2} + \frac{\pi^2}{6}\right)$$

$$I_{\rm c2} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{L^2} - \frac{1}{2}\ln^2\frac{\mu^2}{L^2} + \frac{\pi^2}{6}\right)$$

$$I_s = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{L^2P^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{L^2P^2} + \frac{\pi^2}{6}\right)$$

$$I = I_{\rm h} + I_{\rm c1} + I_{\rm c2} + I_s = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2} + \frac{\pi^2}{3} + O(\lambda)\right)$$

Finite (and correct!)

IR divergences of the hard part are in one-to-one correspondance to UV divergences of the low-energy regions

- True in general: IR divergences of on-shell amplitudes are equal to UV divergences of soft+collinear contributions

The cancellation of divergences involves a nontrivial interplay of soft and collinear log's

$$-\frac{1}{\epsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{L^2}+\frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{L^2P^2}=-\frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2}$$

- Leads to constraints on IR structure of on-shell amplitudes. → last lecture