1. The renormalized quark mass $m_q(\mu)$ in the $\overline{\text{MS}}$ scheme is related to the bare mass m_q through

$$m_q = Z_q\, m_q(\mu) = \left(1 - 3C_F\,\frac{\alpha_s(\mu)}{4\pi}\frac{1}{\epsilon}\right) m_q(\mu)\,.$$

a.) The mass fulfills the RG equation[1]

$$\mu\frac{dm(\mu)}{d\mu} = \gamma_m(\alpha_s)\, m(\mu) = \left(\gamma_m^{(0)}\frac{\alpha_s(\mu)}{4\pi} + \ldots\right) m(\mu)\,. \tag{1}$$

Derive the value of the one-loop anomalous dimension $\gamma_m^{(0)}$ using the magic relation (2) below.

b.) Assume that the value of $m_q(\mu)$ is known for some reference scale $\mu = \mu_0$. Solve the RG equation (1) at one-loop level to obtain the quark mass $m_q(\mu)$ as a function of $m_q(\mu_0)$ and the coupling constants $\alpha(\mu)$ and $\alpha(\mu_0)$. Use

$$\mu\frac{d\alpha_s}{d\mu} = \frac{d\alpha_s}{d\ln\mu} = \beta(\alpha_s) = -2\alpha_s\left(\beta_0\frac{\alpha_s(\mu)}{4\pi} + \ldots\right)$$

to rewrite the integration over the scale

$$\int d\ln\mu = \int\frac{d\alpha}{\beta(\alpha)}$$

as an integral over the running coupling and expand $\beta(\alpha)$ to leading order in α_s. Does the mass $m_q(\mu)$ increase or decrease as one evolves from the reference scale μ_0 to lower scales?

2. In this exercise we derive the "magic relation"

$$\gamma = 2\alpha_s\frac{\partial Z_{[1]}}{\partial\alpha_s} \tag{2}$$

between an anomalous dimension γ of an operators and the $1/\epsilon$-pole of the associated Z factor

$$Z = 1 + \sum_{k=1}^{\infty}\frac{1}{\epsilon^k}Z_{[k]}(\alpha_s),$$

in the $\overline{\text{MS}}$ scheme in dimensional regularization.

[1] In the literature, the RG equation for the mass is often defined with the opposite sign, which flips the sign of the coefficients. Also for the Z-factor of operators different conventions exist.

a.) We first need the β function in d dimensions. To this end, use that $\mu\frac{d}{d\mu}\alpha_s^{(0)} = 0$, with bare coupling $\alpha_s^{(0)} = Z_g^2 \mu^{2\epsilon}\alpha_s(\mu)$ to show

$$\beta(\alpha_s, \epsilon) = -2\epsilon\,\alpha_s - 2\alpha_s Z_g^{-1}\mu\frac{d}{d\mu}Z_g.$$

b.) Now write $\beta(\alpha_s, \epsilon) = \beta(\alpha_s) + \sum_{k=1}^{\infty}\epsilon^k \beta_k(\alpha_s)$, and use $\mu\frac{d}{d\mu}Z_g = \frac{\partial Z_g}{\partial \alpha_s}\beta(\alpha_s, \epsilon)$ to find

$$Z_g\beta(\alpha_s, \epsilon) = -2\epsilon\,\alpha_s Z_g - 2\alpha_s\frac{\partial Z_g}{\partial \alpha_s}\beta(\alpha_s, \epsilon).$$

Expanding this relation at large ϵ, you should find $\beta_1 = -2\alpha_s$, $\beta_k = 0$ for $k > 1$, and the first "magic relation" $\beta(\alpha_s) = 4\alpha_s^2\frac{\partial Z_{1g}}{\partial \alpha_s}$, where Z_{1g} is the first term in the ϵ expansion of Z_g.

c.) Finally, repeat the same strategy for the anomalous dimension encountered in the lecture, which should lead to Eq. (2).