1. The elements of the lie group $SU(N)$ can be represented as

$$U(\{\alpha\}) = \exp(i\,\boldsymbol{T}^a\,\alpha^a)$$

where the matrices $\boldsymbol{T}^a$ are the generators of the group with $a = 1\,\ldots\,N^2 - 1$. The parameters α^a characterize the group element. The multiplication properties of the group elements are encoded in the commutation relations of the generators which take the form

$$[\boldsymbol{T}^a, \boldsymbol{T}^b] = i f^{abc}\,\boldsymbol{T}^c\,. \qquad (1)$$

The constants f^{abc} are called the structure constants of the group.

The representation of the group $SU(N)$ in terms of $N \times N$ matrices is called the fundamental representation $\boldsymbol{T}^a \equiv \boldsymbol{T}^a_F = \boldsymbol{t}^a$. For $N = 3$, the generators are related to the Gell-Mann matrices $\boldsymbol{t}^a = \boldsymbol{\lambda}^a/2$.

a.) Show that the unitarity of the $SU(N)$ matrices entails hermiticity of the generators and that the requirement $\det U(\{\alpha\}) = 1$ implies that the generators have to be traceless.

b.) Show that the structure constants f_{abc} of $SU(N)$ are real and fulfill the Jacobi identity

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \qquad (2)$$

This identity can be obtained by considering the Jacobi identity

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

for the generator matrices $\boldsymbol{T}^a$ and rewriting the commutators in terms of structure constants using their defining relation (1).

c.) Define generator matrices of the adjoint representation as

$$(\boldsymbol{T}^a_A)_{bc} = -i f^{abc}$$

and show that (2) implies that these indeed fulfill the Lie algebra (1).

2. Show that the conjugate matrices of the fundamental representation $\boldsymbol{T}^a_F = \boldsymbol{t}^a$, defined as

$$\boldsymbol{T}^a_{\bar{F}} = -(\boldsymbol{t}^a)^T = -(\boldsymbol{t}^a)^*$$

are a representation of $SU(N)$. What is the conjugate representation of the adjoint representation?

3. For a representation $\boldsymbol{T}_R^a$ of a Lie group, the quantity $\boldsymbol{C}_R = \sum_a \boldsymbol{T}_R^a \boldsymbol{T}_R^a$ is called the quadratic Casimir operator of the representation.

   (a) Show that this quantity commutes with all generators $\left[\boldsymbol{C}_R, \boldsymbol{T}_R^b\right] = 0$. For an irreducible representation, Schur's lemma then implies that the operator is proportional to the unit matrix $\boldsymbol{C}_R = C_R \mathbf{1}$.

   (b) Compute the values of C_F and C_A, the Casimir invariants of the fundamental representation $\boldsymbol{T}_F^a = \boldsymbol{t}^a$ snd and the adjoint representation, respectively, i.e.

$$\boldsymbol{t}^a \boldsymbol{t}^a = C_F \, \mathbf{1} \,, \qquad f^{acd} f^{bcd} = C_A \delta^{ab} \,.$$

   Remember that we normalized

$$\mathrm{Tr}\big(\boldsymbol{t}^a \boldsymbol{t}^b\big) = T_F \, \delta^{ab} = \frac{1}{2} \delta^{ab} \,.$$

For C_A, show first that

$$f^{acd} f^{bcd} = 4 \mathrm{Tr}\big(C_F \boldsymbol{t}^a \boldsymbol{t}^b - \boldsymbol{t}^a \boldsymbol{t}^c \boldsymbol{t}^b \boldsymbol{t}^c\big)$$

and simplify the last term using

$$\boldsymbol{t}_{ij}^a \boldsymbol{t}_{kl}^a = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right), \tag{3}$$

which follows when considering the decomposition of a general $N \times N$ matrix into the unit matrix and $\boldsymbol{t}^a$.

   a.) *Bonus:* Derive the $SU(N)$ Fierz identity (3).