Effective Field Theory — Serie 4

Decoupling in QED and anomalous magnetic moment — 30. 4. 2024

1. Derive the Gordon identity ($q = p_2 - p_1$)

$$\bar{u}(p_2)\,\gamma^\mu\,u(p_1) = \frac{1}{2m}\,\bar{u}(p_2)\,\left[(p_1+p_2)^\mu + i\sigma^{\mu\nu}q_\nu\right]\,u(p_1),$$

by application of the EOM.

2. The contribution of a hypothetical massive photon with propagator

$$\frac{-ig^{\mu\nu}}{q^2 - m_\gamma^2 + i\epsilon} \tag{1}$$

to a_e is given by

$$\Delta a_e = \frac{\alpha}{\pi}\int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)\frac{m_\gamma^2}{m_e^2}}. \tag{2}$$

The x integration arises from Feynman parametizing the relevant loop integral. Show that in the limit $m_\gamma \to 0$ this formula reproduces the Schwinger result $a_e = \alpha/(2\pi)$.

3. Vacuum polarization by a $\mu^+\mu^-$ pair modifies the photon propagator according to

$$\frac{-ig^{\mu\nu}}{q^2+i\epsilon} \to \frac{-ig^{\mu\nu}}{q^2(1-\Pi(q^2))+i\epsilon} = \frac{-ig^{\mu\nu}}{q^2+i\epsilon}(1-\Pi(0)+\bar{\Pi}(q^2)) + \mathcal{O}(\Pi(q^2)^2),$$

where

$$\bar{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0) = \frac{2\alpha}{\pi}\int_0^1 dx\,x(1-x)\log\frac{m_\mu^2 - x(1-x)q^2}{m_\mu^2},$$

and we separated off $1-\Pi(0)$ because this part gets absorbed into the renormalization of the electric charge. We would like to bring the rest into the form (1) to calculate the corresponding contribution to a_e. As a first step, show that

$$\mathrm{Im}\,\Pi(q^2) = -\frac{\alpha}{3}\sqrt{1-\frac{4m_\mu^2}{q^2}}\Big(1+\frac{2m_\mu^2}{q^2}\Big).$$

Hint: the imaginary part comes from the logarithm, with $\log(-|a| \pm i\epsilon) = \log|a| \pm i\pi$ *and* $q^2 \to q^2 + i\epsilon$.

4. We can now write $\Pi(q^2)$ in the form

$$\Pi(q^2) = \Pi(0) + \frac{q^2}{\pi} \int_{4m_\mu^2}^{\infty} ds \frac{\operatorname{Im}\Pi(s)}{s(s - q^2 - i\epsilon)}\,, \tag{3}$$

which has a denominator of the same form as (1) and then use (2) to compute the contribution of the muon loop to the anomalous magnetic moment. Show that

$$\Delta a_e = -\frac{\alpha}{\pi^2} \int_{4m_\mu^2}^{\infty} ds\, \frac{\operatorname{Im}\Pi(s)}{s} \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)\frac{s}{m_e^2}}\,. \tag{4}$$

5. Using (3) one can immediately perform the integral in (4). Show that this leads to

$$\Delta a_e = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \bar{\Pi}\Big(-\frac{x^2 m_e^2}{1-x} \Big) = \frac{1}{45} \frac{m_e^2}{m_\mu^2} \Big(\frac{\alpha}{\pi}\Big)^2 + \mathcal{O}\Big(\frac{m_e^4}{m_\mu^4}\Big),$$

We have thus confirmed that the contribution of the muon is suppressed by m_e^2/m_μ^2, in accordance with the fact that the effect is encoded into a dimension 6 operator in the effective theory.