1.)

$$\bar{u}(p_2)\,\sigma_{\mu\nu} q^\nu\, u(p_1)$$

$$= \frac{i}{2}\,\bar{u}(p_2)\left[\gamma^\mu \not{p}_2 - \gamma^\mu \not{p}_1 - \not{p}_2\gamma^\mu + \not{p}_1\gamma^\mu\right]\bar{u}(p_1)$$

$\uparrow$

$q^\mu = p_2^\mu - p_1^\mu$

$$= \frac{i}{2}\,\bar{u}(p_2)\left[-m\gamma^\mu + 2p_2^\mu - \gamma^\mu m - m\gamma^\mu - \gamma^\mu m + 2p_1^\mu\right]u(p_1)$$

$\uparrow$

use EOM + commutation

$$= i\,\bar{u}(p_2)\left[-2m\gamma^\mu + (p_2^\mu + p_1^\mu)\right]u(p_1) \qquad (*)$$

Compute $\frac{i}{2m}\cdot(*)$:

$$\bar{u}(p_2)\frac{1}{2m}\,i\sigma_{\mu\nu} q^\nu\, u(p_1)$$

$$= \bar{u}(p_2)\gamma^\mu u(p_1) - \frac{p_1^\mu + p_2^\mu}{2m}\,\bar{u}(p_2)\bar{u}(p_1)$$

This is the Gordon identity.

2.) Trivial: For $m_f = 0$

$$\Delta a_e = \frac{\alpha}{\pi}\int_0^1 dx\,(1-x) = \frac{\alpha}{\pi}\int_0^1 dx\,x = \frac{\alpha}{2\pi}.$$

3.)

Imaginary part:

$$\mathrm{Im}\,\ln(-z-i\varepsilon) = -i\pi\,\Theta(z)$$

$$\mathrm{Im}\left[\ln\left(1 - x\bar{x}\frac{q^2}{m^2} - i\varepsilon\right)\right]$$

$$= -i\pi\,\Theta\Big(\underbrace{\bar{x}x}_{\leq \frac{1}{4}}\frac{q^2}{m^2} - 1\Big)$$

Imaginary part only for $q^2 \geq 4m^2$.

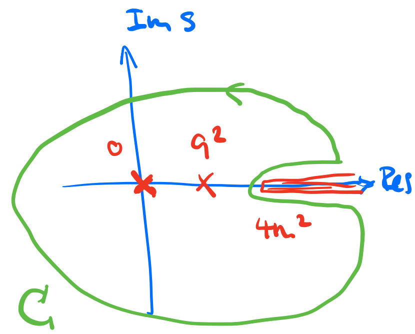
$$\mathrm{Im}\left[\bar{\Pi}(q^2)\right] = \frac{2\alpha}{\pi}\int_{x_-}^{x_+} dx\,x\bar{x}\,(-\pi)$$

$$\left[\; x_\pm = \frac{1}{2} \pm \sqrt{1 - \frac{4m^2}{q^2}} \;\right]$$

$$= -\frac{\alpha}{3\pi}\sqrt{1-\frac{4m^2}{q^2}}\left(1+\frac{2m^2}{q^2}\right)\Theta\left(q^2-4m^2\right)$$

4.) Dispersion relation: Consider

$$f(s) = \frac{\Pi(s)}{s(s-q^2)}$$



$$\oint_{C_1} ds\, f(s) = 2\pi i\left[\frac{\Pi(0)}{-q^2} + \frac{\Pi(q^2)}{q^2}\right]$$

$$= \frac{2\pi i}{q^2}\left[\Pi(q^2) - \Pi(0)\right]$$

$$\rightarrow \quad \Pi(q^2) = \Pi(0) + \frac{q^2}{2\pi i}\int_{C_1}\frac{\Pi(s)}{s(s-q^2)}\,ds$$

Now use that the circle at infinity does not give a contribution. The integral above and below the cut is $2i \cdot \mathrm{Im}(s)$

$$\rightarrow \Pi(q^2) = \Pi(0) + \underbrace{\frac{q^2}{\pi} \int_{4m^2}^{\infty} ds \frac{\operatorname{Im} \Pi(s)}{s(s-q^2)}}_{\bar{\Pi}(q^2)}$$

Which is (3).

$$\Delta a_e = -\frac{\alpha}{\pi^2} \int_{4m_\mu^2}^{\infty} ds \, \frac{\operatorname{Im} \Pi(s)}{s} \cdot \underbrace{\int_0^1 dx \, \frac{x^2(1-x)}{x^2 + (1-x)\, s/m_e^2}}_{\int_0^1 dx \, \frac{m_e^2 x^2}{\underbrace{\frac{m_e^2 x^2}{1-x}}_{-\hat{q}^2} + s}}$$

$$= -\frac{\alpha}{\pi} \int_0^1 dx \, \frac{1}{\hat{q}^2} \bar{\Pi}(\hat{q}^2) \, m_e^2 x^2$$

$$= \frac{\alpha}{\pi} \int_0^1 dx \, \bar{\Pi}\left(-\frac{x^2 m_e^2}{1-x}\right) \frac{1-x}{m_e^2 x^2} m_e^2 x^2$$

$$= \frac{\alpha}{\pi} \int_0^1 dx \, (1-x) \, \bar{\Pi}\left(-\frac{x^2 m_e^2}{(1-x)}\right)$$

$$= \frac{\alpha}{\pi}\left(-\frac{m_e^2}{3}\right)\bar{\Pi}'(0) + O(m_e^4/m_\mu^4)$$

Using the representation of $\bar{\Pi}(q^2)$ from 3.)

we get

$$\bar{\Pi}'(0) = -\frac{2\alpha}{\pi}\int_0^1 dx\, \frac{x^2\bar{x}^2}{m_\mu^2} = -\frac{\alpha}{15 m_\mu^2 \pi}$$

$$\longrightarrow \Delta a_e = \frac{\alpha^2}{\pi^2}\frac{1}{45}\frac{m_e^2}{m_\mu^2} + O\left(\frac{m_e^4}{m_\mu^4}\right).$$