Solutions Exercise 2

1a.) We will analyze all integration-by-part identities of the form

$$\int d^d x \, \partial_\mu f(\phi) = 0$$

boundary term

& $\phi \to 0$ for $x_\mu \to \infty$

Short-hand notation:

$$\partial_\mu f(\phi) \hat{=} 0$$

Using these identities allows us to reduce the number of terms in $\mathcal{L}_{eff}$.

Example: Kinetic term

$$0 \hat{=} \partial_\mu(\phi\,\partial^\mu \phi) \hat{=} \partial_\mu \phi\, \partial^\mu \phi + \phi \Box \phi$$

$$\rightarrow \quad \partial_\mu \phi\, \partial^\mu \phi \hat{=} - \underline{\phi \Box \phi}$$

the only term with two fields and two derivatives

→ only one of the two terms needs to be included in $\mathcal{L}$.

We now repeat the procedure for the dimension six operators suppressed by $1/M^2$. At $d = 6$, we have:

i.) Four derivatives, two fields

ii.) Two derivatives, four fields

iii.) No derivatives, six fields

i.) <u>Four derivatives, two fields</u>

Possible terms: $\phi \Box^2 \phi = O_1$

$\partial_\mu \phi \, \partial^\mu \Box \phi = O_2$

$\Box \phi \, \Box \phi = O_3$

$\partial_\mu \partial_\nu \phi \, \partial^\mu \partial^\nu \phi = O_4$

Identities:

$$0 \hat{=} \partial^\mu (\partial_\mu \phi \, \Box \phi) \hat{=} \Box \phi \, \Box \phi + \partial_\mu \phi \, \partial^\mu \Box \phi$$

$$= O_3 + O_2$$

$$\rightarrow O_3 \hat{=} - O_2$$

$$0 \hat{=} \partial^\mu (\phi \, \partial_\mu \Box \phi) = \partial^\mu \phi \, \partial_\mu \Box \phi + \phi \Box^2 \phi$$

$$= O_2 + O_1$$

$$\rightarrow O_2 \hat{=} - O_1$$

$$0 \hat{=} \partial_\mu \left(\partial_\nu \phi \, \partial^\mu \partial^\nu \phi \right) = O_4 + O_2$$

$$\rightarrow \quad O_4 \hat{=} -O_2$$

Therefore:

$$O_4 \hat{=} O_3 \hat{=} -O_2 \hat{=} \hat{O}_1$$

$\rightarrow$ Only one independent operator!

Can choose O_1.

ii.) <u>Two derivatives, four fields</u>

$$\partial_\mu \phi^n = (n-1) \phi^{n-1} \partial_\mu \phi$$

$\rightarrow$ sufficient to consider derivatives on a single fields

Possible terms: $(\partial_\mu \phi)(\partial^\mu \phi) \, \phi^2 = O_1$

$(\Box \phi) \, \phi^3 = O_2$

$$\partial_\mu (\partial_\mu \phi \; \phi^3) \hat{=} (\Box \phi) \cdot \phi^3 + \partial_\mu \phi \; \partial_\mu \phi^3$$

$$= (\Box \phi) \phi^3 + 3 \partial_\mu \phi \partial^\mu \phi \; \phi^2$$

$$= O_2 + 3 O_1$$

$$\rightarrow \quad O_1 \hat{=} -\frac{1}{3} O_2$$

Only single operator is needed, we can choose O_2. In the lecture, we used $\phi^2 \Box \phi^2 \hat{=} \phi^2 \partial_\mu (2 \phi \partial^\mu \phi)$

$$= 2 \phi^2 \partial_\mu \phi \partial^\mu \phi + 2 \phi^3 \Box \phi$$

$$= 2 O_1 + 2 O_2$$

$$= \frac{4}{3} O_2$$

iii.) <u>Six fields</u>

Only possible term is ϕ^6.

$$\rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\tilde{\lambda}}{4!}\phi^4$$

$$- \frac{C_{2,4}}{2!M^2}\phi\,\Box^2\phi - \frac{C_{4,2}}{4!M^2}\phi^2\Box\phi^2 - \frac{C_{6,0}}{6!M^2}\phi^6$$

is the most general $\mathcal{L}_{eff}$, up to dimension 8 terms.

1b.) Field redefinition

$$\phi_L = \phi_L + \frac{\alpha}{M^2}\Box\phi_L + \frac{\beta}{M^2}\phi_L^3$$

only need to plug this into

$$\mathcal{L}_{eff}^{(4)} = +\frac{1}{2}\partial_\mu\phi_L\partial^\mu\phi_L - \frac{m^2}{2}\phi_L^2 - \frac{\lambda}{4!}\phi_L^4$$

since contributions from other terms are suppressed by $1/M^4$ or more.

$$\mathcal{L}_{eff} \longrightarrow \mathcal{L}_{eff} + \partial_\mu\phi_L\partial^\mu\left(\frac{\alpha}{M^2}\Box\phi_L + \frac{\beta}{M^2}\phi_L^3\right)$$
$$- \tilde{m}^2\phi_L\left(\frac{\alpha}{M^2}\Box\phi_L + \frac{\beta}{M^2}\phi_L^3\right)$$
$$- \frac{\tilde{\lambda}}{3!}\phi_L^3\left(\frac{\alpha}{M^2}\Box\phi_L + \frac{\beta}{M^2}\phi_L^3\right) + O\left(\frac{1}{M^4}\right)$$

Now we can used the identities from 1a.) to bring all terms into the same form as the ones in the original $\mathcal{L}_{eff}$.

$$\mathcal{L}_{eff} = \mathcal{L}_{eff} + \frac{\tilde{m}^2 \alpha}{M^2} \partial_\mu \phi_L \partial^\mu \phi_L - \frac{\tilde{m}^2 \beta}{M^2} \phi_L^4$$

$$- \frac{\alpha}{M^2} \phi_L \Box^2 \phi_L - \frac{\beta}{2M^2} \phi_L^2 \Box \phi_L^2$$

$$- \frac{\tilde{\lambda}}{3!} \frac{3}{4} \frac{\alpha}{M^2} \phi_L^2 \Box \phi_L^2 - \frac{\tilde{\lambda}}{3!} \frac{\beta}{M^2} \phi_L^6$$

$$+ O\left(\frac{1}{M^4}\right)$$

After the transformation, the coefficient of $\phi_L \Box^2 \phi_L$ is

$$\frac{1}{2M^2}\left(-C_{2,4} - 2\alpha\right)$$

so choosing $\alpha = -C_{2,4}/2$ eliminates this term. The coefficient of $\phi^2 \Box \phi^2$ is:

$$\frac{1}{4!M^2}\left(-C_{4,2} - 3\tilde{\lambda}\alpha - \frac{4!}{2}\beta\right)$$

Choice $\beta = -\frac{2}{4!}\left(C_{4,2} + 3\tilde{\lambda}\alpha\right)$

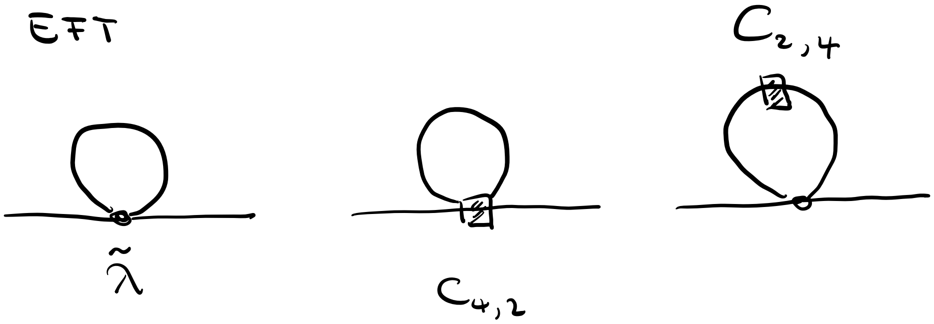
$$= -\frac{1}{12}\left(C_{4,2} - \frac{3}{2}\tilde{\lambda}C_{2,4}\right)$$

eliminates this term.

The remaining terms from the field redefinition can be absorbed into redefinitions of the parameters.

1c.)

EFT

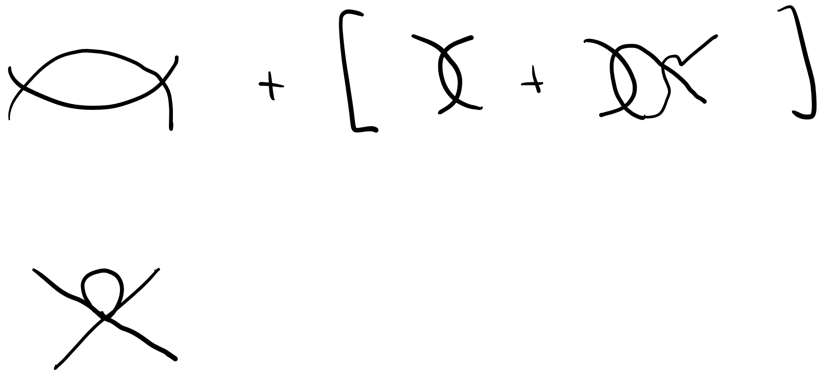


not present if these terms were eliminated....

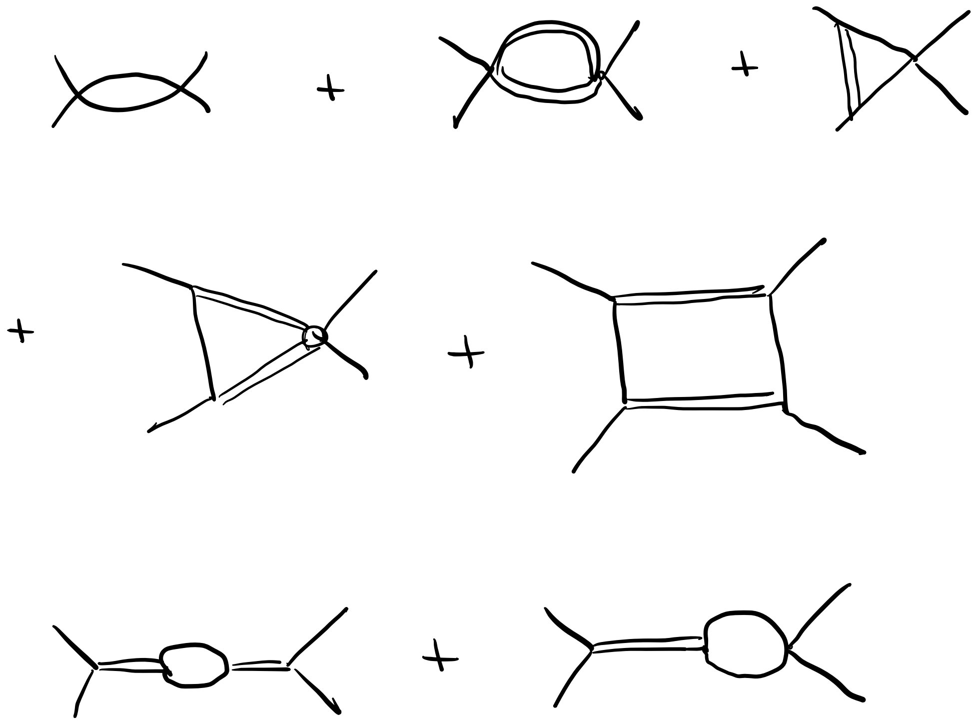
Note: we only draw connected 1PI (one-particle irreducible) diagrams.

Full theory

1d.) EFT (w/o $C_{2,4}$, $C_{4,2}$!)



Full theory



+ permutations of external legs.

2.) To get gauge invariant results, one can work with the field strength tensor and derivatives of it.
(Or equivalently, with products of covariant derivatives.)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad ; \quad [F_{\mu\nu}] = 2.$$

EOM: $\quad \partial^\mu F_{\mu\nu} = j_\nu = 0$ in our case.

a.) Only $d=4$ operator is
$F^{\mu\nu} F_{\mu\nu}$.

b.) Candidates at $d=6$:

$$O_1 = F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\mu = 0$$

(vanishing reflects C-parity $A_\mu \to -A_\mu$)

Because of the EOM, terms of the form $\partial_\mu F^\mu{}_\nu$ etc. can all be eliminated.

This leaves

$$O_2 = \partial_\rho F_{\mu\nu} \partial^\rho F^{\mu\nu}$$

$$O_3 = F_{\mu\nu} \Box F^{\mu\nu} \hat{=} -O_2$$

Jacobi identity: $[D_\rho, [D_\mu, D_\nu]] + \text{cyclic} = 0$

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu}$$

$$\to O_2 = (-\partial_\mu F_{\nu\rho} - \partial_\nu F_{\rho\mu}) \partial^\rho F^{\mu\nu}$$

$$\hat{=} F_{\nu\rho} \partial^\rho \underbrace{\partial_\mu F^{\mu\nu}}_{\hat{=} 0} + F_{\rho\mu} \partial^\rho \underbrace{\partial^\nu F^{\mu\nu}}_{\hat{=} 0} \quad \text{EOM}$$

$$\hat{=} 0 \; !$$

→ No physical $d=6$ operators !

3.) We can either discuss total derivatives or momentum conservation

a.) In momentum space: two fields with $p_1 + p_2 = 0 \rightarrow p_2 = -p_1$.

Invariants:

$$p_1^2 \, , (p_1^2)^2 \, , (p_1^2)^4 \, , \ldots$$

→ only a single operator.

Position space:

$$(\partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_n} \phi)(\partial_{\mu_1} \cdots \partial_{\mu_n} \phi)$$

$$= (-1)^n \phi \, \Box^n \phi$$

b.) Momentum space, n fields $(i,j = 1, \dots n)$

Symmetric quadratic invariants:

$$O_1 = \sum_i p_i^2 \quad ; \quad O_2 = \sum_{i \neq j} p_i \cdot p_j$$

Momentum conservation:

$$O_2 = \sum_i p_i \cdot (-p_i) = -O_1$$

$\rightarrow$ only a single invariant O_1

c.) Quartic invariants

$$O_1 = \sum_i (p_i^2)^2$$

$$O_2 = \sum_{i \neq j} p_i^2 \, p_j^2$$

$$O_3 = \sum_i \sum_{j \neq k} p_i^2 \; p_j \cdot p_k = - \sum_i \sum_j p_i^2 \, p_j^2$$

$$= -O_1 - O_2$$

$$O_4 = \sum_{i \neq j} (p_i \cdot p_j)^2$$

$$O_5 = \sum_{i \neq j} \sum_{k \neq l} (p_i \cdot p_j)(p_k \cdot p_l)$$

$$\leq \left(\sum_i p_i^2\right)\left(\sum_k p_j^2\right) = O_1 + O_2$$

$$O_6 = \sum_{i \neq j \neq k \neq l} p_i \cdot p_j \; p_k \cdot p_l$$

$$= \sum_{i \neq j \neq k} p_i \cdot p_j \,(- p_k \cdot p_i - p_k \cdot p_j - p_k^2)$$

$$= \sum_{i \neq j} p_i \cdot p_j \,(p_i^2 + p_i p_j + p_i \cdot p_j + p_j^2)$$

$$- \sum_k p_k^2 \sum_i p_i^2$$

$$= 2 \sum_i (p_i^2)^2 + 2 \sum_{i \neq j} (p_i \cdot p_j)^2 - \left(\sum_i p_i^2\right)\left(\sum_j p_j^2\right)$$

⇝ Everything linear in a momentum can be eliminated using momentum conservation

⇝ It appears to me that O_1, O_2, O_4 are the only independent structures.