D. Goldstone's theorem

Assume that we have a theory which is invariant under a symmetry group G.

This leads to a set of conserved currents. Consider one such current $J^A_\mu(x)$ and the associated conserved charge

$$Q^A = \int d^3x \; J^A_0(x).$$

Let's assume that $Q^A|0\rangle \neq 0$, i.e. that the symmetry is spontaneously broken.

Let's assume that $H|0\rangle = 0|0\rangle$. It then follows that

$$H Q^A|0\rangle = [H, Q^A]|0\rangle + Q^A H|0\rangle$$
$$= 0$$

→ We have a state $Q^A|0\rangle$ which is degenerate in energy with the vacuum.

Problem: $\| Q^A|0\rangle \|^2 = \langle 0| Q^A Q^A |0\rangle$

$$= \int d^3x \int d^3y \underbrace{\langle 0| Q^A(\vec{x})\, Q^A(\vec{y})|0\rangle}_{F(x-y) \text{ (translation inv.)}} = \infty$$

This by itself is only moderately disturbing, since

$$\langle \vec{p}\,' | \vec{p} \rangle = (2\pi)^3 \omega_p \underbrace{\delta^{(3)}(\vec{p}-\vec{p}\,')}_{\infty \text{ for } \vec{p}=\vec{p}\,'}$$

but the state $Q^A|0\rangle$ is also strange since it does not carry momentum. (What is a massless particle without 3-momentum?)

A clean way to identify particles is to look at the Fourier transform of the two-point function of two local operators. A $\frac{1}{p^2}$ - pole indicates the presence of massless particles.

Let's assume that we have a set of scalar fields ϕ_a with the right quantum numbers of the Goldstone bosons (GBs) so that

$$[Q^A, \phi_a] = (t^A)_{ab} \phi_b$$

For a fundamental scalar field, this follows from

$$J^{\mu A} = i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} t^A_{ab} \phi_b$$

and the canonical commutation relations, but our field ϕ_a can be a general, composite object.

We further assume that

$$\langle 0 | \phi_a(0) | 0 \rangle \neq 0 \quad \text{for some } a.$$

and will now show that this implies the presence of massless bosons.

Following Goldstone, Salam and Weinberg, we analyze

$$\langle 0 | [J^A_\mu(y), \phi_a(x)] | 0 \rangle$$

$$e^{iPx} \phi_a(0) e^{-iPx}$$

$$= \sum_X \Big[\langle 0 | J^A_\mu(y) | X \rangle \langle X | \phi_a(x) | 0 \rangle - \langle 0 | \phi_a(x) | X \rangle \langle X | J^A_\mu(y) | 0 \rangle \Big]$$

$$= \sum_X \Big[e^{ip_X(x-y)} \langle 0 | J^A_\mu(0) | X \rangle \langle X | \phi_a(0) | 0 \rangle - e^{-ip_X\cdot(x-y)} \langle 0 | \phi_a(0) | X \rangle \langle X | J^A_\mu(0) | 0 \rangle \Big]$$

$$= \int d^4p \sum_X \delta^{(4)}(p - p_X) [\ldots]$$

Define:

$$\sum_X \delta^4(p-p_X)\, \langle 0|\, J^A_\mu(0)\,|X\rangle\langle X|\, \phi_a(0)\,|0\rangle$$

$$= i p_\mu\, \Theta(p^0)\, \rho^A_a(p^2)$$

$\uparrow$ spectral density

$$\sum_X \delta^{(4)}(p-p_X)\, \langle 0|\, \phi_a(0)\,|X\rangle\langle X|\, J^A_\mu(0)\,|0\rangle$$

$$= i p_\mu\, \Theta(p^0)\, \tilde{\rho}^A_a(p^2)$$

For $x^0 = y^0$ and $|\vec{x}-\vec{y}| > 0$ (spacelike separation)

the commutator should vanish, which implies

$\rho^A_a(p^2) = -\tilde{\rho}^A_a(p^2)$. [To see it, do $\vec{p} \to -\vec{p}$

in the second term.]

$$\langle 0|\,[\, J^A_\mu(y), \phi_a(x)\,]\,|0\rangle = -\frac{\partial}{\partial y^\mu} \int d^4p\, \Theta(p^0)$$

$$\underbrace{\left[e^{ip(x-y)}\, \rho^A_a(p^2) + e^{-ip(x-y)}\, \tilde{\rho}^A_a(p^2) \right]}_{\left[e^{ip(x-y)} - e^{-ip(x-y)} \right] \rho^A_a(p^2)}$$

By introducing $1 = \int d\mu^2\, \delta(p^2-\mu^2)$, we finally have the following spectral representation

$$\langle 0|\,[\, J_\mu^A(y)\, ,\, \Phi_a(x)\,]\,|0\rangle$$

$$= -\frac{\partial}{\partial y^\mu} \int d\mu^2\, \rho_a^A(\mu^2)\, \Delta(x-y)$$

with $\Delta(z) = \int d^4p\, \delta(p^2-\mu^2)\, \theta(p^0)\left[e^{ipz} - e^{-ipz}\right]$

causal propagator of a free, massive scalar field!

$$\Delta(z) = 0 \quad \text{for} \quad z^2 < 0$$

$$\neq 0 \quad \text{for} \quad z^0 > 0$$

Now we use that $\partial_\mu J^\mu = 0$:

$$\rightarrow\ 0 \overset{!}{=} -\Box_y \int d\mu^2\, \rho_a^A(\mu^2)\, \Delta(x-y)$$

$$= \int d\mu^2\, \rho_a^A(\mu^2)\, \mu^2\, \Delta(x-y)$$

$$\left[\ \Box\, e^{\pm ipy} = -p^2\, e^{\pm ipy}\right.$$

Since $\Delta(z) \neq 0$ for $z^2 > 0$, we must have

$$\rho_a^A(\mu^2)\mu^2 = 0 \qquad (\rho_a^A(\mu^2) \geq 0!)$$

$\rho_a^A(\mu^2)$ is a distribution, so this relation implies

$$\rho_a^A(\mu^2) = C_a^A \cdot \delta(\mu^2)$$

If $c \neq 0$, the correlation function contains states with mass $\mu^2 = 0$! To fix c, consider $x^0 = y^0 = 0$ and

$$\to \langle 0 | [J_0^A(y), \phi_a(x)] | 0 \rangle$$

$$= 2i \int d\mu^2\, \rho_a(\mu^2) \int d^4p\, \theta(p^0)\, \delta(p^2 - \mu^2)\, p^0 e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$= i \int d\mu^2\, \rho_a(\mu^2) \underbrace{\int d^3p\, e^{i\vec{p}(\vec{x}-\vec{y})}}_{(2\pi)^3 \delta(\vec{x}-\vec{y})}$$

$$\rightarrow \int d^3y \, \langle 0 | [J_0^A(y), \phi_a(x)] | 0 \rangle$$

$$= \langle 0 | [Q^A, \phi_a(x)] | 0 \rangle$$

$$= i(2\pi)^3 \int d\mu^2 \, \rho_a^A(\mu^2) = i(2\pi)^3 \, C_a^A$$

But we assumed

$$[Q^A, \phi_a] = t_{ab}^A \phi_b$$

$$\rightarrow \quad i(2\pi)^3 C_a^A = t_{ab}^A \langle 0 | \phi_b(0) | 0 \rangle$$

$$\langle 0 | \phi_b(0) | 0 \rangle \neq 0 \implies C_a^A \neq 0$$

$\implies$ states with mass $\mu = 0$.