C. QCD Lagrangian and Feynman rules

Quantum Chromo-Dynamics (QCD) is a non-abelian gauge theory. These theories are generalizations of QED. QED is based on the gauge symmetry

$$\psi(x) \to e^{i\alpha(x)}\,\psi(x).$$

While the fermion mass term $\Delta\mathcal{L} = m\,\bar\psi(x)\psi(x)$ is invariant under this symmetry, the usual kinetic term is not because

$$\bar\psi(x)\, i\not\partial\, \psi(x) \to \bar\psi\, e^{-i\alpha(x)}\, i\not\partial\, e^{i\alpha(x)}\psi$$

$$= \bar\psi i\not\partial\psi \underbrace{-i(\partial_\mu\alpha)}\,\bar\psi\gamma^\mu\psi\,.$$

To define an invariant kinetic term, one introduces a gauge field transforming as

$$\underset{\uparrow\ \text{convention.}}{e}A^\mu(x) \to eA^\mu - i\partial^\mu\alpha\,.$$

The covariant derivative

$$iD_\mu\psi = (i\partial_\mu - eA_\mu)\psi \to e^{i\alpha(x)}\, iD_\mu\psi$$

transforms in the same way as $\psi(x)$ so that $\bar{\psi} i \not{D} \psi$ is gauge invariant. Note that

$$[iD_\mu, iD_\nu]\psi = -ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi$$
$$= -ie F_{\mu\nu} \psi$$

is not a derivative on ψ, but just a function of the gauge field. The Lagrangian of QED is

$$\mathcal{L}_{QED} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi .$$

The construction of the QCD Lagrangian follows exactly the same steps, except that one considers a fermion field with three components and the symmetry group SU(3) instead of U(1), i.e. gauge transformations

$$\psi(x) \equiv \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \longrightarrow U(x) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \begin{array}{l} \text{"red"} \\ \text{"green"} \\ \text{\& "blue"} \end{array}$$

$U(x) \in SU(3)$, i.e. $UU^\dagger = \mathbb{1}$, $\det(U) = 1$.

The matrix $U(x)$ can be parameterized as

$$U(x) = \exp\left[i \alpha^a(x) t^a \right]$$

where $a = 1, \dots, 8$ is summed over. The matrices t^a are the <u>generators</u> of $SU(3)$. (For $SU(N)$, there are N^2-1 generators) The generators fulfill the algebra

$$[t^a, t^b] = i f^{abc} t^c$$

structure constants.

An explicit representation $t^a = \frac{\lambda^a}{2}$ are the Gell-Mann matrices λ^a*. The construction of a gauge invariant Lagrangian proceeds in complete analogy to the QED case. One defines a covariant derivative

$$iD_\mu \psi = [i\partial_\mu + g_s A^a_\mu t^a] \psi$$

one for each transformation α^a

* http//en.wikipedia.org/wiki/Gell-Mann_Matrices

After a gauge transformation $U(x)$, we get

$$i D_\mu \psi \to \Big[U(x)\, i\partial_\mu + g A'^a_\mu t^a U(x) + i(\partial_\mu U(x)) \Big] \psi(x) \overset{!}{=} U(x) D_\mu \psi$$

$$\to A'^a_\mu t^a = U(x) A^a_\mu t^a U^{-1}(x) - \frac{i}{g} (\partial_\mu U(x)) U^{-1}(x)$$

The non-abelian field-strength tensor is

$$F_{\mu\nu} = \frac{1}{ig} [i D_\mu, i D_\nu] = F^a_{\mu\nu} t^a$$

$$= \partial_\mu A^a_\nu t^a - \partial_\nu A^a_\mu t^a - ig [A^b_\mu t^b, A^c_\nu t^c]$$

$$= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) t^a - ig \underbrace{A^b_\mu A^c_\nu i f^{bca}}_{!} t^a$$

$F_{\mu\nu} \to U(x) F_{\mu\nu} U^{-1}(x)$ under gauge transformations.

The gauge action is

$$\mathcal{L} = -\frac{1}{4T_F}\, tr\left[F^{\mu\nu} F_{\mu\nu} \right]$$

$$= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

with $tr\left[t^a t^b \right] = T_F\, \delta^{ab}$. It is conventional to use $T_F = \frac{1}{2}$.

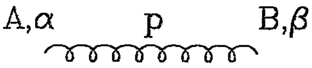
The main difference to QED is that the field strength contains a term quadratic in the gauge field. The gauge action $(F^a_{\mu\nu})^2$ thus contains cubic and quartic terms in A^μ, i.e. gluon self-interactions.

Explicitly, one has

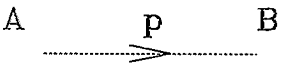
$$\mathcal{L}_{QCD} = -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right)$$
$$+ \bar{\Psi}_\alpha \left(i\not{\partial} - m + g\, t^a_{\alpha\beta} \not{A}^a \right) \Psi_\beta$$
$$- \frac{g}{2} f^{abc} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) A^{b\mu} A^{c\nu}$$
$$- \frac{g^2}{4} f^{abc} f^{cde} A_\mu^a A_\nu^b A^{e\mu} A^{d\nu}$$

By Fourier transforming (and symmetrizing) the action one obtains the QCD Feynman rules. There is one complication: as in the case of QED, one needs to fix the gauge to perform perturbative calculations. The gauge fixing terms are more complicated than for QED. In addition to a term $\Delta\mathcal{L} = -\frac{1}{2\xi}\left(\partial_\mu A^a\right)^2$ one needs a ghost field to represent the so-called Faddeev-Popov determinant.

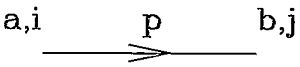
This ghost field is unphysical but needed to cancel unphysical polarization states of the gluon field. In the computations we perform in this lecture we do not encounter such ghost diagrams, but the Feynman rules that follow include them for completeness.



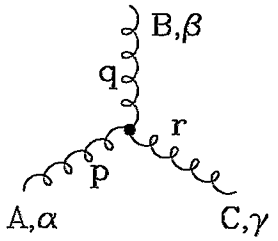
$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^{\alpha}p^{\beta}}{p^{2}+i\epsilon}\right]\frac{i}{p^{2}+i\epsilon}$$



$$\delta^{AB}\frac{i}{(p^{2}+i\epsilon)}$$ (ghost propagator)

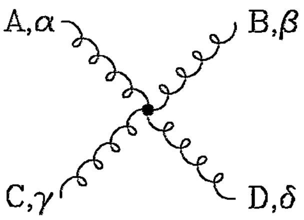


$$\delta^{ab}\frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

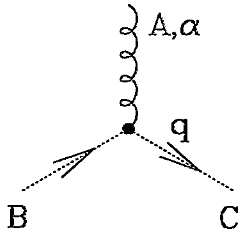


$$+g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

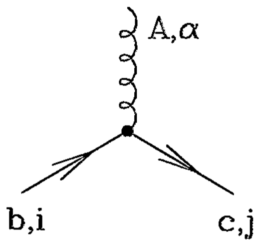
(all momenta incoming, $p+q+r = 0$)



$$-ig^{2}\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^{2}\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^{2}\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$



$$-g\ f^{ABC}q^{\alpha}$$ (ghost - gluon vertex)



$$+ig\ (t^{A})_{cb}\ (\gamma^{\alpha})_{ji}$$

(from Keith Ellis)
(but with $g \to -g$)