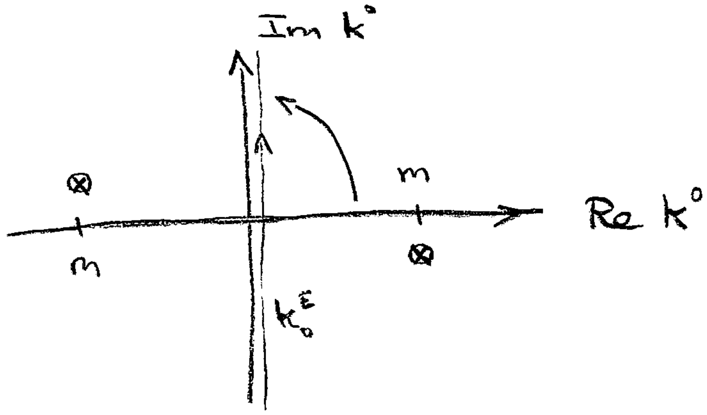
Loop integrals in dimensional regularization

want to derive the formula

$$I = \underbrace{\int d^d k \, \frac{(k^2)^\alpha}{(k^2 - m^2 + i\varepsilon)^\beta}}_{\text{well defined for } d/2 + \alpha > 0 \text{ and } \beta - \alpha - d/2 > 0} = i \, \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \cdot (m^2)^{d/2 + \alpha - \beta} \, (-1)^{\alpha+\beta} \times \frac{\Gamma(\alpha + d/2)\,\Gamma(\beta - \alpha - d/2)}{2\Gamma(\beta)}$$

As a second step, we'll show that all one-loop integrals can be brought into this form.

Perform Wick rotation $k^0 = i k^0_E$ $\quad \vec{k} = \vec{k}_E$



$$I = i \int d^d k_E \, \frac{(-k_E^2)^\alpha}{(-k_E^2 - m^2)^\beta} = i (-1)^{\alpha+\beta} \int d^d k \, \frac{(k^2)^\alpha}{(k^2 + m^2)^\beta}$$

Drop "E" label

$$\int d^d k \, \frac{(k^2)^\alpha}{(k^2+m^2)^\beta} = \Omega_d \int_0^\infty dk \, \frac{k^{d-1+2\alpha}}{(k^2+m^2)^\beta}$$

solid angle
of sphere in d-dim.

$[k^2 = m^2 x]$

$$= \frac{1}{2}\int_0^\infty dx \, \frac{x^{d/2+\alpha-1}}{(x+1)^\beta} (m^2)^{d/2+\alpha-\beta} \, \Omega_d$$

$$= \frac{\Omega_d}{2} (m^2)^{d/2+\alpha-\beta} \int_0^1 dy \, y^{d/2+\alpha-1} (1-y)^{\beta-\alpha-\frac{d}{2}-1}$$

$$\left[\; y = \frac{x}{1+x} = 0 \ldots 1 \; ; \; x = \frac{y}{1-y} \qquad dx = \frac{1}{(1-y)^2} dy \right.$$

$$\int_0^1 dy \, y^{a-1} (1-y)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{\Omega_d}{2} (m^2)^{d/2+\alpha-\beta} \frac{\Gamma(d/2+\alpha)\Gamma(\beta-\alpha-\frac{d}{2})}{\Gamma(\beta)}$$

Almost done. The last ingredient we need is Ω_d.

A neat (and standard) trick to get it, is to use

$$\int_{-\infty}^{\infty} d^d x \, e^{-\vec{x}^2} = \left[\int_{-\infty}^{\infty} dx \, e^{-x^2} \right]^d = (\sqrt{\pi})^d$$

$$\int_{-\infty}^{\infty} d^d x \, e^{-\vec{x}^2} = \Omega_d \int_0^{\infty} dx \, x^{d-1} e^{-x^2}$$

$$= \frac{\Omega_d}{2} \int_0^{\infty} dy \, y^{d/2} e^{-y} = \frac{\Omega_d}{2} \Gamma\left(\frac{d}{2}\right)$$

$$\Longrightarrow \quad \Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \, ;$$

d	1	2	3	4
Ω_d	2	2π	4π	$2\pi^2$

So we have shown that

$$\boxed{\int d^d k \, \frac{(k^2)^\alpha}{(k^2 - m^2 + i\varepsilon)^\beta} = i(-1)^{\alpha+\beta} \pi^{d/2} (m^2)^{d/2 + \alpha - \beta} \times \frac{\Gamma(\alpha + d/2)\,\Gamma(\beta - \alpha - d/2)}{\Gamma(\beta)\,\Gamma(d/2)}}$$

The LHS is only defined for $\alpha + d/2 > 0$ (IR conv.)

$\beta - \alpha - d/2 > 0$ (UV convergence). The result for the integral is an analytic function of d, α and β except for poles at

$$\alpha + d/2 = 0, -1, -2, -3, \ldots$$

$$\beta - \alpha - d/2 = 0, -1, -2, \ldots$$

since $\Gamma(x)$ has poles at $x = 0, -1, -2, \ldots$

We now <u>define</u> the integral in dim. reg. as the RHS. Choosing $d = 4 - 2\varepsilon$, our integrals are well defined for $\varepsilon \neq 0$. To take the limit $\varepsilon \to 0$

one expands

Euler-γ: 0,577...

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left(\frac{1}{\varepsilon} - \gamma_E + \underbrace{1 + \ldots + \frac{1}{n}}_{H_n}\right) + O(\varepsilon)$$

As a last step, we need to show that all loop integrals can be brought into the form

$\int d^d k \, \frac{(k^2)^\alpha}{(k^2 - m^2)^\beta}$. This is achieved using the Feynman parameterization

$$\int_0^1 dx \, \frac{1}{[xA + (1-x)B]^2} = \frac{1}{AB}$$

to combine all propagators into a single one.

Let's look at an example:

$$S^{\mu\nu} = \int d^d k \, \frac{k^\mu k^\nu}{(k^2 + i\varepsilon)((k-p)^2 + i\varepsilon)}$$

$$= \int d^d k \int_0^1 dx \, \frac{k^\mu k^\nu}{[\underbrace{(1-x)k^2 + xk^2}_{k^2} - 2xpk + xp^2]^2}$$

$$= \int d^d k \int_0^1 dx \, \frac{k^\mu k^\nu}{[(k - xp)^2 + x(1-x)p^2 + i\varepsilon]^2}$$

Now shift $k \to k + xp$

$$S^{\mu\nu} = \int d^dk \int_0^1 dx \; \frac{k^\mu k^\nu + x^2 p^\mu p^\nu + x(p^\mu k^\nu + k^\mu p^\nu)}{[\,k^2 - M^2\,]^2}$$

$$M^2 = -p^2 x(1-x) + i\varepsilon$$

The terms linear in k vanish upon integration.
For the $k^\mu k^\nu$ terms, we use Lorentz invariance:

$$\int d^dk \; k^\mu k^\nu f(k^2) = \int d^dk \; g^{\mu\nu} \tilde{f}(k^2)$$

$$\to \int d^dk \; k^2 f(k^2) = \int d^dk \; d \, \tilde{f}(k^2)$$

$$\to \quad \tilde{f} = \frac{k^2}{d} f(k^2)$$

$$\Rightarrow \quad S^{\mu\nu} = \int_0^1 dx \int d^dk \; \frac{\frac{1}{d} g^{\mu\nu} k^2 + x^2 p^\mu p^\nu}{[k^2 - M^2]^2}$$

In this form, we can now use our master formula for the k-integration.

For more complicated integrals, the parameterization

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \int_0^1 dx_1 \cdots \int dx_n \, \delta(1 - \sum_{i=1}^{n} x_i)$$

$$\times \frac{\prod_i x_i^{m_i - 1}}{\left[\sum_i x_i A_i \right]^{\sum m_i}} \frac{\Gamma(m_1 + \cdots + m_n)}{\Gamma(m_1) \cdots \Gamma(m_n)} \qquad (*)$$

is useful.

Proof: Take derivatives of $\int dx \int dy \, \delta(1-x-y) \frac{1}{[xA + yB]^2}$ with respect to A and B to get

$$(**) \qquad \frac{1}{A^m B^n} = \int dx \int dy \frac{x^{m-1} y^{n-1}}{[xA + yB]^{m+n}} \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \delta(1-x-y)$$

Then prove (*) by induction: have $n = 2$, which is just (**). Assume $n-1$ has the form (*) then use (**) to combine A_n with the rest.

$$\frac{1}{A_1^{m_1}\cdots A_{n-1}^{m_{n-1}} A_n^{m_n}} = \int_0^1 dy \int_0^1 dx\, \delta(1-x-y)$$

$$* \int_0^1 dx_1 \cdots dx_{n-1} \prod_i x_i^{m_i-1} \frac{\Gamma(m_1+\cdots+m_{n-1})}{\Gamma(m_1)\cdots\Gamma(m_{n-1})} \delta\Big(1-\sum^{n-1} x_i\Big)$$

$$\cdot \frac{\Gamma(m_1+\cdots+m_n)}{\Gamma(m_n)\Gamma(m_1+\cdots+m_{n-1})} x^{m_n-1} y^{m_1+\cdots+m_{n-1}-1} \frac{1}{[xA_n+yB]^{\sum^n m_i}}$$

$$= \int_0^\infty dy \int_0^\infty dx_m\, \delta(1-x_m-y)\,\delta\Big(1-\sum^{n-1} x_i\Big) \prod_i^n x_i^{m_i-1}$$

$$\cdot \frac{\Gamma(m_1+\cdots+m_n)}{\Gamma(m_1)\cdots\Gamma(m_n)} \frac{y^{m_1+\cdots+m_{n-1}-1}}{(xA_n+y(x_1A_1+\cdots x_nA_n))]^{\sum_i^n m_i}}$$

Now rescale $x_i \to x_i/y$ for $i = 1 \ldots m-1$ and integrate over y to derive the induction step.