4.4.3 Effective Lagrangian

Now that we know the transformation of the Goldstone bosons, it is straightforward to write down the effective Lagrangian in the chiral limit $m_q = 0$. After this, we'll have to implement the symmetry breaking terms which involve the quark masses.

Under a chiral transformation $U \to V_R U V_L^+$ and we want to find an effective Lagrangian $\mathcal{L}_{eff}(U)$ which is invariant under chiral transformations. Since $U(x)$ is dimensionless, the terms with higher powers of $U(x)$ are unsuppressed, so we order terms by derivatives

$$\mathcal{L}_{eff} = \underbrace{f_0(U)}_{O(1)} + \underbrace{f_1(U)\,\Box U}_{O(p^2)} + \underbrace{f_2(U)\,\partial_\mu U \partial^\mu U}_{O(p^2)}$$

$$+ \ \dots$$

* Chiral symm. implies $f_0(U) = f_0(V_R U V_L^+)$.

Choose $V_R = 1$, $V_L = U$. $\Rightarrow$ $f_0(U) = f_0(1) =$ const.

$\Rightarrow$ $O(1)$ terms are an irrelevant constant. Drop.

* The f_1-term can be aborbed into f_2 using integration by part

$$\int d^4x \; f_1(U)\,\Box U = -\int d^4x \; f_1'(U)\,\partial_\mu U \,\partial^\mu U$$

$$\Rightarrow \mathcal{L}_{eff} = f(U)\,\partial_\mu U\,\partial^\mu U .$$

$$= \tilde{f}(U)\,\Delta_\mu\,\Delta^\mu \qquad \text{with} \quad \Delta_\mu = (\partial_\mu U)U^+$$

The quantity Δ_μ transforms as $\Delta_\mu \to V_R\,\Delta_\mu\,V_R^+$ and is invariant under V_L transformations.

$$\mathcal{L}_{eff} = \tilde{f}(U V_L^+)\,\Delta_\mu\,\Delta^\mu$$

$$\underset{V_L = U.}{\overset{}{\underset{\uparrow}{=}}} \tilde{f}(1)\,\Delta_\mu\,\Delta^\mu$$

The last question is how the indices of the matrices Δ_μ are contracted. The only possibility is

$$\mathcal{L}_{eff} = c \cdot \mathrm{tr}[\Delta_\mu\,\Delta^\mu]$$

[Mathematically, this amounts to the question how one can form a singlet from two adjoint representations of SU(N) and indeed there is only a single possibility corresponding to the trace.
]

$$\mathrm{tr}[\Delta^\mu \Delta_\mu] = \mathrm{tr}\left[(\partial_\mu U)U^+(\partial^\mu U)U^+\right]$$

$$\underset{\substack{\uparrow \\ \partial_\mu(UU^+)=0}}{=} -\mathrm{tr}[\partial_\mu U\, U^+ U \partial_\mu U^+] = -\mathrm{tr}[\partial_\mu U \partial^\mu U^+]$$

$$\mathcal{L}_{eff} = \frac{F^2}{4}\,\mathrm{tr}\left[\partial_\mu U \partial^\mu U\right] + O(p^4)$$

The prefactor has been chosen to get canonically normalized kinetic terms for the π's. To see this we now expand

$$U(x) = \exp\left[\frac{i}{F}\vec{\pi}\cdot\vec{\sigma}\right] = \mathbb{1} + \frac{i}{F}\vec{\pi}\cdot\vec{\sigma} - \frac{1}{2F^2}\vec{\pi}^2\,\mathbb{1}$$

$$\mathcal{L}_{eff} = \frac{1}{2}\partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi} + \frac{1}{48F^2}\,\mathrm{tr}\left\{[\partial_\mu\pi,\pi][\partial^\mu\pi,\pi]\right\}.$$

The effective Lagrangian has several remarkable properties:

1.) One parameter F determines all π-interactions

2.) Symmetry requires interactions with arbitrary many pions.

3.) Derivative couplings: the interactions vanish if the momenta go to zero.

Our effective Lagrangian is only valid in the limit $m_q = 0$ and we should now also implement the quark mass terms which break the symmetry,

$$\mathcal{L}_m = - \bar{q}_R \mathcal{M} q_L - \bar{q}_L \mathcal{M}^+ q_R$$

$$\text{with } \mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}.$$

Note that $\mathcal{L}_m$ would be invariant if $\mathcal{M}$ transformed as $\mathcal{M} \to V_R \mathcal{M} V_L^+$. This property can now be used to construct $\mathcal{L}_{eff}(U, m)$: one treats m as an external source which transforms as $m \to V_R m V_L$. $\mathcal{L}_{eff}$ must then be invariant as well. Expanding in m, the lowest invariant term is

$$\mathcal{L}_{s.b} = \frac{F^2 B_0}{2} \operatorname{tr}\left[m U^+ + m^+ U \right]$$

This term gives a mass to the π's. For $SU(2)$

$$\mathcal{L}_{s.b} = \frac{F^2 B_0}{2} \operatorname{tr}[m] \left(-\frac{1}{F^2} \vec{\pi}^2 \right)$$

$$= -\frac{B_0}{2} (m_u + m_d) \vec{\pi}^2 = -\frac{M_\pi^2}{2} \vec{\pi}^2 .$$

$\Rightarrow$ The masses of the π's are equal and <u>proportional</u> to the <u>sum</u> $m_u + m_d$.

To relate the quantity B_0 to a QCD matrix element treat m as an external source $m \equiv [m(x)]_{ij}$ and then take a functional derivative of the full and effective theory partition function.

$$\frac{1}{i}\frac{\delta}{\delta m_{ij}(x)} Z_{QCD} = -\langle 0 | \bar{q}_{L,i}(x)\, q_{R,j}(x) + \bar{q}_{R,j}(x)\, q_{L,i}(x) | 0 \rangle$$

$$\frac{1}{i}\frac{\delta}{\delta m_{ij}(x)} Z_{eff} = \frac{F^2 B_0}{2} \langle 0 | (U^+)_{ji}(x) + U_{ij}(x) | 0 \rangle$$

The classical action is minimized by $\vec{\pi} = 0$, $U = \mathbb{1}$.

Up to pion loop corrections, we thus have

$$\Rightarrow \quad F^2 B_0\, \delta_{ij} = -\langle 0 | \bar{q}_{L,i}\, q_{R,j} + \bar{q}_{R,j}\, q_{L,i} | 0 \rangle$$

$$F^2 B_0 = -\langle 0 | \bar{u}u | 0 \rangle = -\langle 0 | \bar{d}d | 0 \rangle$$

$\Rightarrow$ B_0 corresponds to the quark condensate in the limit $m_q \to 0$.

So $$M_\pi^2 = \underbrace{(m_u + m_d)}_{\substack{\text{explicit} \\ \text{breaking.}}} \underbrace{\left(\frac{-\langle 0 | \bar{u}u | 0 \rangle}{F^2} \right)}_{\substack{\text{spontaneous} \\ \text{breaking}}} + O(m_q^2)$$

Since $p_\pi^2 = M_\pi^2 \propto m_q$, the quark masses count as $O(p^2)$.

The three π's have the same mass because the quadratic term in $U(x) = \mathbb{1} + \frac{i}{F}\vec{\pi}\vec{\sigma} - \frac{1}{2F^2}\vec{\pi}^2 \cdot \mathbb{1} + \ldots$ is proportional to the unit matrix since $\frac{1}{2}\{\sigma^i, \sigma^j\} = \delta^{ij}\mathbb{1}$.

In the SU(3) case $\{\lambda^a, \lambda^b\}$ is nontrivial and one finds

$$M_\pi^2 = (m_u + m_d)B + O(m_q^2)$$

$$M_{K^\pm}^2 = (m_u + m_s)B + O(m_q^2)$$

$$M_{K^0}^2 = (m_d + m_s)B + O(m_q^2)$$

$$M_\eta^2 = \frac{1}{3}(m_u + m_d + 4m_s)B + O(m_q^2)$$

(Gell-Mann, Oakes, Renner '68)

$M_K^2 \sim 13 \times M_\pi^2$ because $m_s \gg m_u, m_d$.

$$M_\pi^2 - 4M_K^2 + 3M_\eta^2 = 0 + O(m_q^2)$$

(Gell-Mann - Okubo formula.)

To understand how the mesons interact with photons, W- and Z-bosons, it is useful to introduce external sources with the appropriate quantum numbers both in the full and the effective theory.

For QCD, we add

$$\mathcal{L}_{QCD} = \mathcal{L}_0 + \mathcal{L}_1 ,$$

$$\mathcal{L}_0 = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \bar{q} i \not{D} q ,$$

$$\mathcal{L}_1 = v^a_\mu V^\mu_a + a^a_\mu A^\mu_a - s^a S_a - p^a P_a ,$$

$$V^\mu_a = \bar{q} \gamma^\mu \frac{\lambda_a}{2} q \; ; \; A^\mu_a = \bar{q} \gamma^\mu \gamma_5 \frac{\lambda^a}{2} q \; ;$$

$$S_a = \bar{q} \frac{\lambda_a}{2} q \; ; \; P_a = \bar{q} i \gamma_5 \frac{\lambda_a}{2} q .$$

We also include singlets via $\lambda_0 = \sqrt{\frac{2}{3}} \mathbb{1}$.

The external fields $v^a_\mu(x)$, $a^a_\mu(x)$, $s^a(x)$, $p^a(x)$ can be used to probe different aspects of QCD. Quark masses are included in $s^a(x)$.

To construct $\mathcal{L}_{eff}$ in the presence of these sources, one can use the fact that $\mathcal{L}_{QCD}$ becomes invariant under <u>local</u> transformations

$$q_L(x) \to V_L(x) q_L(x) \; ; \; q_R(x) \to V_R(x) q_R(x)$$

provided the external fields transform like gauge fields:

$$r_\mu = (v_\mu + a_\mu) \longrightarrow V_R (v_\mu + a_\mu) V_R^+ - i(\partial_\mu V_R) V_R^+$$

$$\ell_\mu = (v_\mu - a_\mu) \longrightarrow V_L (v_\mu - a_\mu) V_L^+ - i(\partial_\mu V_L) V_L^+$$

$$(s + ip) \longrightarrow V_R (s + ip) V_L^+$$

where $v_\mu = v_\mu^a \frac{\lambda^a}{2}$, etc.

It is easy to construct a locally invariant effective Lagrangian. At leading power, it is sufficient to replace ∂_μ by the covariant derivative:

count as $O(p)$

$$i D_\mu U = i\partial_\mu U + (v_\mu + a_\mu) U - U (v_\mu - a_\mu)$$

so that

$$\mathcal{L}_{eff} = \frac{F^2}{4} \mathrm{tr}\left[D_\mu U D^\mu U^+ \right] + \frac{F^2 B}{2} \mathrm{tr}\left[\chi U^+ + \chi^+ U \right] + O(p^4)$$

with $\chi = s + ip$.

$O(p^2)$

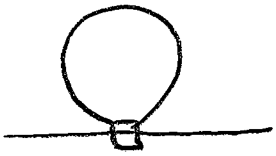
At $O(p^4)$ $\mathcal{L}_{eff}$ has the form (Gasser + Leutwyler '84, '85)

$$
\begin{aligned}
\mathcal{L}^{(4)} = {} & \frac{l_1}{4}\left(tr\left[D_\mu U D^\mu U^+\right]\right)^2 + \frac{l_2}{4}\, tr\left[D_\mu U D_\nu U^+\right] \\
& * \, tr\left[D^\mu U D^\nu U^+\right] + \frac{l_3}{4}\left(tr\left[\chi U^+ + U \chi^+\right]\right)^2 \\
& + \frac{l_4}{4}\, tr\left[D_\mu \chi D^\mu U^+ + D_\mu U D^\mu \chi^+\right] \\
& + \dots
\end{aligned}
$$

For $SU(3)$ $\mathcal{L}^{(4)}$ has 12 coupling constants,

for $SU(2)$ 10. low energy constants.

Let us note that one-loop graphs from $\mathcal{L}^{(2)}$ are of order p^4. So to obtain results to this order one needs tree-level diagrams from $\mathcal{L}^{(4)}$ as well as the one-loop corrections generated by $\mathcal{L}^{(2)}$. E.g.



$\mathcal{L}^{(2)}$

$$\propto \int d^4k \, \frac{1}{k^2 - M_\pi^2}\, k^2 \propto M_\pi^4$$

There is one complication: $\mathcal{L}_{QCD}$ and $\mathcal{L}_{eff}$ are invariant under local chiral transformations but the partition function

$$Z[v,a,s,p] = \int \mathcal{D}q\,\mathcal{D}\bar{q}\,\mathcal{D}A_\mu\, e^{i\int d^4x\, \mathcal{L}_0 + \mathcal{L}_1}$$

$$= e^{iS_{eff}[v,a,s,p]}$$

is <u>not</u> invariant if the external sources are non-zero because of anomalies in the fermion determinant.

Since the effective theory does not involve fermion fields, invariance of $\mathcal{L}_{eff}$ leads to invariance of the partition fuction. To correct this mismatch, one needs to add to $\mathcal{L}_{eff}$ a term which reproduces the change of the QCD partition function. This term is called the Wess-Zumino-Witten term. $\mathcal{L}_{WZW}$. The full effective theory Lagrangian is

$$\mathcal{L}_{eff} = \mathcal{L}_{inv} + \mathcal{L}_{WZW}.$$

The WZW terms are $O(p^4)$ and do not involve any low-energy constants. In contrast to $\mathcal{L}_{inv}$, the terms in $\mathcal{L}_{WZW}$ contain odd number of GB fields. In particular it contains a term describing an interaction of two vector fields with a π^0, which leads to

$$\Gamma_{\pi^0 \to \gamma\gamma} = \frac{\alpha^2 N_c^2 M_{\pi^0}^3}{192\pi^3 F_\pi^3} \cdot (e_u + e_d)^2 = 7{,}6\,eV$$

The good agreement with the exp. value (7.7 ± 0.6)eV is usually sold as evidence for $N_c = 3$. However, Bär and Wiese '01 pointed out that $e_u + e_d = \frac{1}{N_c}$ for N_c colors so that the rate does not depend on N_c.

In the exercises, we computed $\Gamma_{\pi^\pm} = 2{,}5 \cdot 10^{-8}\,eV$. The π^0 decay is <u>much</u> faster because it is mediated by strong instead of weak interactions.