4.4.2. Transformation properties of Goldstone bosons

In order to construct the most general effective Lagrangian, we need to know how the Goldstone boson fields $\vec{\pi}$ transform under chiral symmetry. Usually fields transform linearly, as a representation of a group $\vec{\varphi} \longrightarrow M(g)\vec{\varphi}$. For Goldstone bosons, the symmetry is realized non-linearly, as we will now see.

Let us consider first the general case of a group G which breaks spontaneously to a subgroup H. There are then $n = n_G - n_H$ Goldstone bosons which we collect into an n-dim vector $\vec{\pi}(x)$. A realization of the group is a mapping

$$\vec{\pi} \longrightarrow \vec{\pi}' = \vec{f}(g, \vec{\pi})$$

for any $g \in G$.

This mapping must obey the composition law

$$\vec{f}(g_1, \vec{f}(g_2, \vec{\pi})) = \vec{f}(g_1 g_2, \vec{\pi})$$

[In general $\vec{f}$ is not a representation, since it is not linear $\vec{f}(g, \lambda\vec{\pi}) \neq \lambda \vec{f}(g, \vec{\pi})$.]

Remarkably, this property determines $\vec{f}$ essentially uniquely. To see this consider the image of the origin $\vec{f}(g, \vec{\pi}=0)$. The elements $h \in H$ map the origin onto itself $\vec{f}(h, 0) = 0$ since H is linearly realized. Moreover

$$\vec{f}(gh, 0) = \vec{f}(g, 0) \qquad \forall h \in H$$

So $\vec{f}$ lives on the coset space G/H. It maps an element of G/H into the space of pion fields. Furthermore it is also invertible, since $\vec{f}(g_1, 0) = \vec{f}(g_2, 0)$ implies $g_1 H = g_2 H$.

Proof: $\vec{f}(e,0) = 0 = \vec{f}(g_1^{-1} g_1, 0)$

$$= \vec{f}(g_1^{-1}, \vec{f}(g_1, 0)) = \vec{f}(g_1^{-1}, \vec{f}(g_2, 0))$$

$$= \vec{f}(g_1^{-1} g_2, 0) = 0$$

$$\Rightarrow g_1^{-1} g_2 \in H \to g_1 H = g_2 H .$$

So the function $\vec{f}(g, 0)$ provides a one-to-one mapping between the coset space G/H and the values of the π field. The transformation of the field follows from the action of $g \in G$ on the coset space. The only freedom left is the choice of coordinates on G/H.

Let us now consider $G = SU_L(2) \times SU_R(2)$ $= \{(V_L, V_R), L \in SU(2), R \in SU(2)\}$ and $H = \{(V, V), V \in SU(2)\}$. The coset space associated with an element $\tilde{g}$ is the set

$$\tilde{g} H = \{(\tilde{V}_L V, \tilde{V}_R V), V \in SU(2)\} .$$

To parametrize G/H, we select one element of each set $\tilde{g}H$. A possible choice is $U = \tilde{V}_R \tilde{V}_L^+$, since

$$(\tilde{V}_L V, \tilde{V}_R V) = (1, \tilde{V}_R \tilde{V}_L^+)\, \underbrace{(\tilde{V}_L V, \tilde{V}_L V)}_{\in H}$$

The transformation law of U under G is

$$U \to V_R U V_L^+$$

for $g = (V_L, V_R)$.

All that is left is to parametrize $U(x) \in SU(2)$. One can use the standard parametrization

$$U(x) = \exp\left[i \frac{\sigma^a \pi^a}{F} \right].$$

$$= \exp\left[i \frac{1}{F} \begin{pmatrix} \pi^0 & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 \end{pmatrix} \right]$$

In the second line, we have rewritten π^1, π^2, π^3 in terms of the linear combinations with definite electric charge. The factor F was introduced to obtain a dimensionless exponent, but it will corresponds to the π decay constant.

Note that we could choose a different parameterization, e.g. $U(x) = \sqrt{1 - \vec{\pi}^2/F^2} + i\vec{\sigma}\cdot\vec{\pi}$.

The π-fields of the two different parametrizations are related by a field redefinition. We have shown earlier, that such transformations leave the physics unchanged.

For SU(3), the standard parametrization is

$$U(x) = \exp\left[\frac{i}{F}\lambda^a\pi^a\right] = \exp\left[\frac{i}{F}\begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{3}\eta \end{pmatrix}\right]$$

To understand, why the field is parameterized in this way, one needs to consider the quark-mass term and the coupling to photons.