4.4. Chiral Perturbation Theory (CHPT)

Let us finally turn to the strong interaction at low energies. Instead of quarks and gluons, the observed particles are hadrons, i.e. mesons such as π's, K's, η, η', ρ, ... and baryons p, n, Δ, Σ, ... The effective Lagrangian is then a function of hadron fields. As in all our previous applications, one starts by writing down the most general $\mathcal{L}_{eff}$ compatible with the symmetries of the underlying theory, i.e. QCD. In contrast to previous examples, we will be unable to perform matching computations due to our limited ability to perform QCD computations at low E (using lattice simulations, it is becoming possible to some extent.) At first sight, it looks like an effective theory of hadrons will be not very predictive since the Wilson coefficients are not known. However, it turns out that chiral symmetry severly constrains the interactions of the light hadrons, and the effective theory approach is very useful to derive the consequences of this approximate symmetry.

4.4.1. Chiral symmetry

Since we will work at very low energies, we can integrate out the heavy-quark flavors and use

$$\mathcal{L}_{QCD}^{eff} = \sum_{f=u,d,s} \bar{q}_f (i\not{D} - m_q) q - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \mathcal{O}\left(\frac{1}{m_Q^2}\right)$$

The theory simplifies further in the <u>chiral limit</u> $m_q \to 0$. Since only the mass term distinguishes different flavors, a flavor symmetry arises. In fact, the symmetry group is even larger: split $q = q_L + q_R = P_L q + P_R q$, with $P_L = \frac{1}{2}(1-\gamma_5)$; $P_R = \frac{1}{2}(1+\gamma_5)$. Then

$$\mathcal{L}_{QCD}^{eff} = \sum_f \Big[\bar{q}_{L,f}\, i\partial\!\!\!/\, q_{L,f} + \bar{q}_{R,f}\, i\partial\!\!\!/\, q_{R,f} + m_f\left(\bar{q}_{L,f}\, q_{R,f} + \bar{q}_{R,f}\, q_{L,f}\right)\Big] - \frac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu}$$

(Note: $\bar{q}_L = q^+ P_L^+ \gamma^0 = q^+ P_L \gamma^0 = \bar{q} P_R$.)

So in the absence of a mass term, $\mathcal{L}$ is invariant under the chiral transformations

$$q_L = \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \longrightarrow V_L \cdot \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = V_L q_L$$

$$q_R \longrightarrow V_R q_R$$

where V_L and V_R are unitary 3×3 matrices.

[Note that V_L and V_R are <u>global</u> rotations in flavor space, while the gauge transformations act in color space.]

Instead of $m_u, m_d, m_s \to 0$, it is also useful to just consider the two-flavor chiral limit $m_{u,d} \to 0$, m_s fixed. In this case, the symmetry transformations are

$$\begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \to V_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}$$

and the transformations can be parametrized as

$$V_{L,R} = \exp\left[i\alpha_{L,R} + i \frac{\sigma^a}{2} \alpha^a_{L,R} \right]$$

the Pauli matrices $\frac{\sigma^a}{2}$ are the generators of SU(2).

For the three-flavor case, the generators are the Gell-Mann matrices $\frac{\lambda^a}{2}$, $a = 1 \dots 8$. We can then consider infinitesimal transformations and Noether's theorem gives a classically conserved current for each transformation $j_\mu \propto \frac{\delta \mathcal{L}}{\delta(\partial^\mu \psi)} \cdot \delta\psi$

$$L_\mu = \bar{q}_L \gamma_\mu q_L, \quad L^a_\mu = \bar{q}_L \gamma_\mu \frac{\lambda^a}{2} q_L,$$

$$R_\mu = \bar{q}_R \gamma_\mu q_R, \quad R^a_\mu = \bar{q}_R \gamma_\mu \frac{\lambda^a}{2} q_R.$$

Instead of left- and right-handed currents, it is convenient to use axial and vector currents:

$$V^\mu = L^\mu + R^\mu = \bar{q}\gamma^\mu q$$
$$A^\mu = R^\mu - L^\mu = \bar{q}\gamma^\mu\gamma_5 q$$

It turns out that A^μ is anomalous, i.e. $\partial_\mu A^\mu \neq 0$ due to quantum effects. More precisely

$$\partial_\mu A^\mu = \frac{N_c g_s^2}{32\pi^2}\,\varepsilon_{\mu\nu\rho\sigma}\,G^{a\mu\nu}G^{a\rho\sigma}. \qquad (N_c = 3.)$$

The remaining $SU_L(3)\times SU_R(3)\times U_V(1)$ transformations are symmetries of the quantum theory. With each current, we can associate a conserved charge

$$Q_V^a = \int d^3x\,\bar{q}\gamma^0\frac{\lambda^a}{2}q$$
$$Q_A^a = \int d^3x\,\bar{q}\gamma^0\gamma_5\frac{\lambda^a}{2}q$$

The $2\times 8+1$ charges Q_V, Q_V^a, Q_A^a commute with the Hamiltonian H_0 of massless QCD

$$[Q_V^a, H_0] = [Q_A^a, H_0] = [Q_V, H_0] = 0.$$

The question is then whether the spectrum of the theory is symmetric, or whether the symmetry is spontaneously broken. Vafa and Witten '84 have shown that the vector-like symmetries are unbroken $Q_V^a |0\rangle = 0$. For the axial symmetry, the situation is more complicated. Let's discuss the two possibilities

A.) Unbroken symmetry $Q_A^a |0\rangle = 0$

$\rightarrow$ Spectrum contains degenerate multiplets of $G = SU_L(3) \times SU_R(3)$.

B.) Spontaneously broken symmetry $Q_A^a |0\rangle \neq 0$

$\rightarrow$ Multiplets of $SU_V(3) \subset G$

$\rightarrow$ Goldstone bosons

Goldstone theorem:

$$H Q_A^a |0\rangle = Q_A^a H |0\rangle = 0$$

Since $Q_A^a |0\rangle \neq 0 \Rightarrow \exists\, 8$ states with $E = 0$.

So we end up with 8 (one for each broken generator) massless, parity-odd, spin 0 states.

Unfortunately, the above argument has a flaw:

$$\langle 0 | Q_A^a Q_A^b | 0 \rangle = \int d^3x \int d^3y \langle 0 | A_0^a(x) A_0^b(y) | 0 \rangle$$
$$= \int d^3x \int d^3y \underbrace{F(x-y)}_{\text{transl. inv.}} = \infty$$

So the "states" $Q_A^a |0\rangle$ have infinite norm.

A rigorous proof is obtained by analyzing the correlation function

$$\langle 0 | [Q_A^a(t), P^a(t, \vec{y})] | 0 \rangle \quad \left(\begin{array}{c} \text{no summation} \\ \text{over } a \end{array} \right)$$

Inserting a basis of states and using current conservation one can show that if the above matrix element is nonvanishing, then the theory contains a massless particle with the same quantum numbers as

$$P^a = \bar{q} \frac{\lambda^a}{2} \gamma_5 q .$$

The above matrix element can be simplified using the equal-time anti-commutation relations

$$\{ q_{\alpha,r}(t,\vec{x}) \, , \, q^{+}_{\beta,s}(t,\vec{y}) \} = \delta_{\alpha\beta}\,\delta_{rs}\,\delta^3(\vec{x}-\vec{y}),$$

$$\{ q_{\alpha,r}(t,\vec{x}), q_{\beta,s}(t,y) \} = 0,$$

$$\{ q^{+}_{\alpha,r}(t,\vec{x}), q_{\beta,s}(t,y) \} = 0.$$

the commutator has the form

$$\{q,q\}=0$$

$$[ab, cd] = a\{b,c\}d - ac\{b,d\} + \{a,c\}db - c\{a,d\}b.$$

$$\{q^+,q^+\}=0.$$

so we get

$$[A^a_0(\vec{x},0), P^a(\vec{y},0)]$$

$$= q^+(\vec{x})\,\gamma^5 \frac{\lambda^a}{2}\,\delta(x-y)\,\gamma^0\gamma^5 \frac{\lambda^a}{2}\, q(\vec{y})$$

$$- q^+(\vec{y})\,\gamma^0\gamma^5 \frac{\lambda^a}{2}\,\delta(x-y)\,\gamma^5 \frac{\lambda^a}{2}\, q(\vec{x})$$

$$= -\frac{1}{2}\,\delta(\vec{x}-\vec{y})\,\bar{q}(\vec{y})\,(\lambda^a)^2 q(\vec{y})$$

Because of $SU_V(3)$ invariance of $|0\rangle$ one further has

$$\langle 0|[Q_A^a, P^a(y)]|0\rangle = \frac{1}{8}\sum_a \langle 0|[Q_A^a, P^a(y)]|0\rangle$$

$$= -\frac{1}{8}\cdot\frac{1}{2}\cdot\frac{16}{3}\langle 0|\bar{q}(0)\,q(0)|0\rangle$$

where we used $\sum_a (\lambda^a)^2 = \frac{16}{3}\mathbb{1}_{3\times 3}$. Also, because of $SU(3)$

$$\underbrace{\langle 0|\bar{q}q|0\rangle}_{\text{quark condensate}} = \langle 0|\bar{u}u + \bar{d}d + \bar{s}s|0\rangle = 3\langle 0|\bar{u}u|0\rangle$$

quark condensate $\bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L$ breaks chiral symmetry.

A nonvanishing quark condensate implies that chiral symm. is spontaneously broken and that there are 8 pseudoscalar Goldstone bosons.

Since the quark masses are non-zero, chiral symmetry is not an exact symmetry of QCD. On the other hand, the u-, d- and s-quark masses are small, so one can treat the mass term of QCD as a perturbation.

Looking at the spectrum, one finds that three mesons $\pi^\pm, \pi^0$ are quite light, $m_\pi \approx 140$ MeV, and nearly degenerate. Since they also are parity-odd and have spin zero, it is plausible that they are the $SU_V(2)$ triplet of "Goldstone" bosons associated with the spontaneous breaking of chiral symmetry in the $\begin{pmatrix} u \\ d \end{pmatrix}$ sector:

$$SU_L(2) \times SU_R(2) \to SU_V(2).$$

Since the small mass-term breaks the symmetry explicitly, they acquire a small mass.

For this reason, they are called pseudo Goldstone bosons.

The lowest lying eight mesons are

$$\pi^+, \pi^-, \pi^0, K^+, K^0, \bar{K}^0, K^0, K^-, \eta$$

again, they have $J^P = 0^-$, and so fit the pattern of symmetry breaking for

$$SU_L(3) \times SU_R(3) \to SU_V(3).$$

If chiral symmetry were unbroken, one would expect multiplets of the full symmetry group: for each parity-odd meson, there should be a (nearly) degenerate parity-even partner.

From these considerations, and from the fact that CHPT is very successful in describing the low-energy phenomenology of QCD, one concludes that chiral symmetry is indeed spontaneously broken.