4. The Standard Model at low energies

We now construct low-energy effective theories for the interactions of the Standard Model (SM).

We'll first treat electromagnetism (Euler Heisenberg Lagrangian), then the weak interaction (Fermi theory) and finally the strong interaction (Chiral Perturbation Theory (CHPT)).

4.1. Euler Heisenberg Theory

Let's consider QED

$$\mathcal{L}_{QED}[A_\mu, \psi] = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{D} - m_e)\psi$$

where $iD_\mu = i\partial_\mu - eA_\mu$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{e}[iD_\mu, iD_\nu]$$

A_μ is the electromagnetic potential and ψ the electron field.

Note that $\mathcal{L}_{QED}$ is the most general renormalizable Lagrangian for an electron interacting with the photon field. In the SM, there are many other heavier charged particles (quarks, W's, μ, τ) but according to EFT logic, the contributions of all the other fields only appear via $1/M$ suppressed operators (where $M = m_\tau, m_{\pi^+}, \dots$). In other words, $\mathcal{L}_{QED}$ is the leading-power effective Lagrangian describing the interaction of $e^\pm$ and γ, and it will be appropriate as long as $E \ll 100\,\text{MeV} \approx m_\mu \sim m_\pi$.

Many practical applications only involve photons at even smaller energies $E \ll 2m_e$. In this case, electron-positron pairs appear only as virtual corrections and we can integrate them out, i.e. construct an effective theory involving only photons.

There is an interesting complication: electron number is conserved, so if we consider a state with $5e^-$, these will be there even as $E \to 0$. To describe this situation correctly, one has to use nonrelativistic effective field theory. We'll get to this later. For now we will just describe these electrons as an external current and add a term

$$\mathcal{L}_J = -e A_\mu J^\mu_{e.m.}$$

to the Lagrangian. This description should work for macroscopic charged objects, as long as we do not excite higher energy levels in their interaction with the photon. Note that this interaction is only consistent if $\partial_\mu J^\mu_{e.m.} = 0$: Under gauge transformations

$A^\mu \to A^\mu + \partial^\mu \phi$, so

$$\int d^4x\, A_\mu J^\mu_{e.m.} \to \int d^4x\, A_\mu J^\mu - \int d^4x\, \phi \underbrace{\partial_\mu J^\mu}_{\stackrel{!}{=} 0}.$$

Examples of configurations fulfilling $\partial_\mu J^\mu = 0$ are

* static charge dist. $J^\mu = (\rho(r), 0)$.
* static current $J^\mu = (0, \vec{J}(r))$ with $\vec{\nabla}\vec{J} = 0$.

If we now consider low-energy photons in the background of a source J^μ, we should be able to describe their interactions in terms of

$$\mathcal{L}_{eff}[A_\mu, J_\mu] = \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \mathcal{L}^{(8)} + \dots$$

The leading-power Lagrangian is

$$\mathcal{L}^{(4)} = -\frac{2}{4} F^{\mu\nu} F_{\mu\nu} - e A_\mu J^\mu$$

and describes free photons. Let us now construct the operators of dimension 6 and 8, whose effects are suppressed by $1/m_e^2$ and $\frac{1}{m_e^4}$.

To obtain $\mathcal{L}_{eff}$, one writes down all possible terms of given dimension. The number of terms can be reduced to a minimal set using

(i) Symmetries, e.g. charge conjugation

$$e \to -e\,,\quad A_\mu \to -A_\mu \quad \Rightarrow F_{\mu\nu} \to -F_{\mu\nu}$$

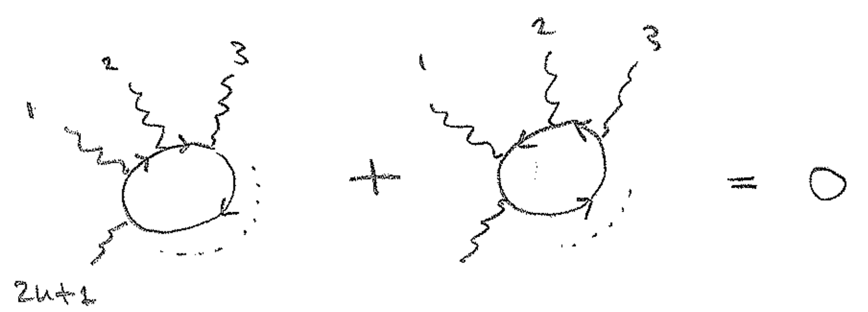
(ii) Properties of $F_{\mu\nu}$, e.g.

$$F^{\mu\nu} = -F^{\nu\mu}$$

$$\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0 \quad \text{(Bianchi identity)}$$

iii) The leading-power EOM $\partial_\mu F^{\mu\nu} = j^\nu$.

Let us construct the $d=6$ terms. Because of the $e \to -e$, $F^{\mu\nu} \to -F^{\mu\nu}$ symmetry, $\mathcal{L}_{eff}$ must be even in $F^{\mu\nu}$. This is the EFT equivalent of Furry's theorem, which states that $2n+1$ photon amplitudes vanish in QED.



This leaves us with terms of the form $\partial^2 F^2$.

Using integration by part, we can always achieve that derivatives are not contracted with the field strength on which they act. This leaves two possible terms:

$$O_1 = F^{\mu\nu} \Box F_{\mu\nu}$$

$$O_2 = (\partial^\rho F^{\mu\nu})(\partial_\mu F_{\rho\nu})$$

Let's use the Bianchi-identity on O_2

$$O_2 = (\partial^\rho F^{\mu\nu})\left[-\partial_\rho F_{\nu\mu} - \partial_\nu F_{\mu\rho}\right]$$

$$= -\partial^\rho\left(F^{\mu\nu}\partial_\rho F_{\nu\mu}\right) + F^{\mu\nu}\Box F_{\mu\nu}$$

$$- \partial^\rho F^{\nu\mu}\partial_\nu F_{\rho\mu}$$

$$= -\partial^\rho(\ldots) + F^{\mu\nu}\Box F_{\mu\nu} - O_2$$

So the two terms O_1 and O_2 are equivalent, $2O_2 \mathrel{\hat{=}} O_1$.

In addition, we can write down terms

$$O_3 = J_\mu J^\mu \quad , \quad O_4 = \partial_\mu F^{\mu\nu} j_\nu .$$

since j^μ has dimension $d=3$. Using integration by part $O_2 \hat{=} \partial_\mu F^{\mu\nu} \partial^\rho F_{\rho\nu}$.

Using the EOM $\partial_\mu F^{\mu\nu} = j^\nu$, our final result is

$$\mathcal{L}^{(6)} = \frac{c_0}{m_e^2} J^\mu J_\mu .$$

This corresponds to a contact interaction between the source and is irrelevant for photon propagation or scattering.

The first terms involving photons appear for $d=8$:

$$\mathcal{L}^{(8)} = \frac{c_1}{m_e^4} \left(F^{\mu\nu} F_{\mu\nu}\right)^2 + \frac{c_2}{m_e^4} F_{\mu\nu} F^{\nu\sigma} F_{\sigma\rho} F^{\rho\mu}$$

In four space-time dimensions, we can rewrite

$$F_{\mu\nu} F^{\nu\sigma} F_{\sigma\rho} F^{\rho\mu} = \frac{1}{4}\left(F^{\mu\nu} \tilde{F}_{\mu\nu}\right)^2 + \frac{1}{2} F^{\mu\nu} F_{\mu\nu}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$.

However, since $\varepsilon^{\mu\nu\rho\sigma}$ is only defined in $d=4$, it is preferable not to use this relation.

⌈ To derive it, use:

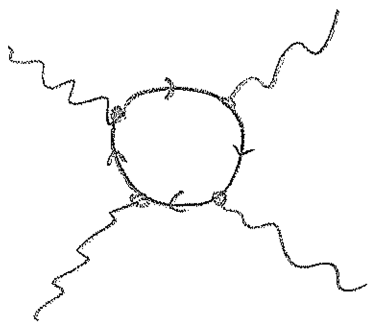
$$\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\,\varepsilon^{\nu_1\nu_2\nu_3\nu_4} = -\begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \cdots \\ \vdots & & \\ \delta^{\nu_1}_{\mu_4} & \cdots & \delta^{\nu_4}_{\mu_4} \end{vmatrix}$$ ⌋

Expressed in terms of $\vec{E}$ and $\vec{B}$

$$F^{\mu\nu}F_{\mu\nu} = -2\,(\vec{E}^2 - \vec{B}^2)$$

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = -4\,\vec{E}\cdot\vec{B} \quad ✓$$

The two terms in $\mathcal{L}^{(8)}$ describe four-photon interactions, but since they are suppressed by $1/m_6^4$, these will be very weak. In QED, these interactions arise from fermion loops



but before evaluating these diagrams let's consider the low-energy $\gamma\gamma \to \gamma\gamma$ scattering cross section. The cross section is

$$d\sigma \sim \left(\frac{1}{m^4}\right)^2 \cdot E^6 \cdot e^2 .$$

$$\uparrow$$

$$\propto \alpha^2$$

The factor E^6 is there to get the correct dimension $d\sigma \sim \frac{1}{E^2}$. An explicit computation yields the unpolarized cross section:

$$\frac{d\sigma_{\gamma\gamma}}{d\Omega} = \frac{1}{4\pi^2}\left(48 c_1^2 + 40 c_1 c_2 + 11 c_2^2\right) \frac{E_{c.m.}^6}{m_e^8}$$

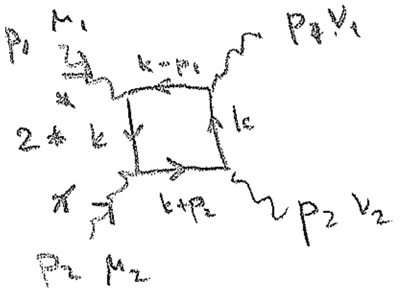
$$\times \left(3 + \cos^2\theta\right)^2$$

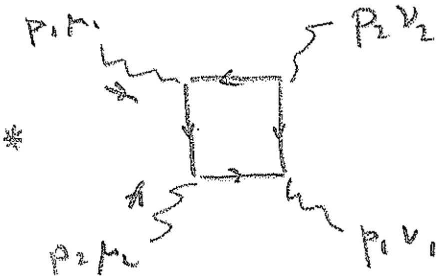
where $\cos\theta$ is the scattering angle in the c.m.s.

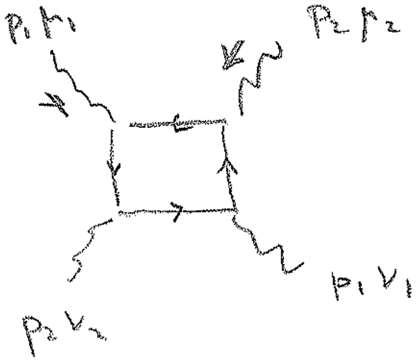
So far $\gamma\gamma$ scattering for $E < m_e$ has not been observed experimentally, but there are plans to measure it using intense lasers (see e.g. 0909.4663).

To determine the Wilson coefficients c_1, c_2, we need to perform a matching computation. It is simplest to consider the $\gamma\gamma$ scattering amplitude. Since we only want to extract two numbers, it is good enough to evaluate the forward amplitude $\gamma(p_1) + \gamma(p_2) \to \gamma(p_1) + \gamma(p_2)$ and to consider two different helicity configurations.

In QED, the amplitude is

$\mathcal{A}^{\mu_1\mu_2\nu_1\nu_2} = 2 *$ 

$+\ 2 *$ 

$+\ 2 *$ 

The factor of two arises because an identical contribution is obtained from the diagram with reversed fermion flow  vs .

To get the scattering amplitude, one has to contract with polarization vectors.

$$\mathcal{A} = \mathcal{A}^{\mu_1\mu_2\nu_1\nu_2}\, \varepsilon_{\mu_1} \varepsilon_{\mu_2} \varepsilon^*_{\nu_1} \varepsilon^*_{\nu_2}$$

For matching purposes, we can consider e.g.

$$\mathcal{A}_1 = g^{\mu_1\mu_2} g^{\nu_1\nu_2} \mathcal{A}_{\mu_1\mu_2\nu_1\nu_2}$$

$$\mathcal{A}_2 = g^{\mu_1\nu_1} g^{\mu_2\nu_2} \mathcal{A}_{\mu_1\mu_2\nu_1\nu_2}$$

in both the full and the effective theory and then solve for c_1, c_2. The computation can be further simplified by expanding the QED diagrams in the small external momenta. This can be done on the integrand level in this case. After this, the necessary integrals all have the form

$$\int d^dk \, \frac{(k^2)^\alpha}{(k^2 - m_e^2)^\beta}$$

and are obtained directly from Appendix A.

The result is $C_1 = -\frac{1}{36}\alpha^2$, $C_2 = \frac{7}{90}\alpha^2$.

That it has to be finite (and that the "renormalized" result does not contain $\ln\left(\frac{m_e^2}{\mu^2}\right)$) can also be understood directly in the EFT: the only loop in the EFT from $\mathcal{L}^{(8)}$ is

$= 0$

$\mathcal{L}^{(18)}$

since a second vertex from $\mathcal{L}^{(8)}$ would give an additional $\frac{1}{m^4}$ suppression and $\mathcal{L}^{(4)}$ and $\mathcal{L}^{(6)}$ do not contain interactions. The operators in $\mathcal{L}^{(8)}$ are thus not renormalized.

Pluging in C_1 & C_2 into our earlier result for the cross section, one has

$$\frac{d\sigma}{d\Omega} = 139\left(\frac{\alpha}{180\pi}\right)^2 \left(3+\cos^2(\theta)\right)^2 \frac{E_\gamma^6}{m_e^8}.$$