3.5. Renormalization-group improved perturbation theory

The Wilson coefficients C_i (the "coupling constants") in $\mathcal{L}_{eff}$ depend on the coupling constants of the full theory as well as the large energy scale M. The dependence on M is logarithmic. Schematically, we have

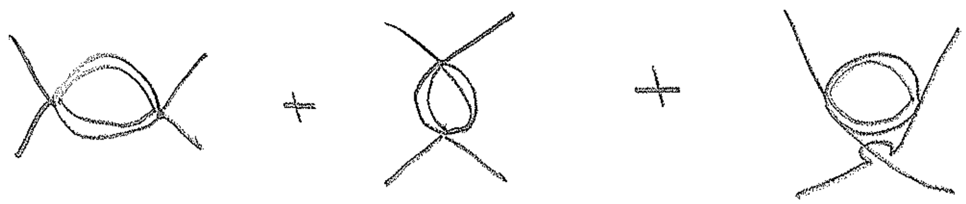
$$C_i(M,\mu,\lambda) = C_i^{(0,0)} + \lambda(\mu)\left[C_i^{(1,1)} \ln\left(\frac{M^2}{\mu^2}\right) + C_i^{(1,0)} \right]$$
$$+ \lambda^2(\mu)\left[C_i^{(2,2)} \ln^2\left(\frac{M^2}{\mu^2}\right) + C_i^{(2,1)} \ln\left(\frac{M^2}{\mu^2}\right) + C_i^{(2,0)} \right]$$
$$+ \dots$$

where the coefficients $C_i^{(n,m)}$ $(m \le n)$ are pure numbers, determined by the matching calculation.

In our matching calculation for $\hat{m}$ we found exactly this structure, except that our full theory has several different couplings.

let's consider $\tilde{\lambda}$ as another example.

At the one-loop level, we encounter contributions



$$= -\frac{3\lambda_{HL}^2}{32\pi^2}\left(-\frac{1}{\varepsilon} + \ln\left(\frac{M^2}{\mu^2}\right) + c\right)$$

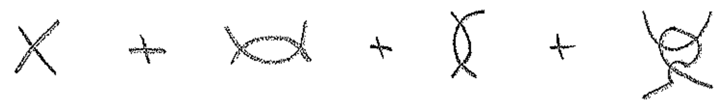
divergence can be absorbed into λ

The finite part generates a matching correction to $\tilde{\lambda}$:

$$\tilde{\lambda}(\mu) = \lambda(\mu) + \frac{3\lambda_{HL}}{32\pi^2}\left[\ln\left(\frac{M^2}{\mu^2}\right) + C\right]$$

The form of the result makes it obvious that we should choose $\mu \approx M$, otherwise perturbation theory will not work well because the $\ln\left(\frac{M^2}{\mu^2}\right)$ terms become large.

On the other hand, let's look at a computation in the EFT. For the $2 \to 2$ amplitude at leading power, we get

$\mathcal{A} =$ 

$$= -\tilde{\lambda}_0\left[1 + \frac{3\tilde{\lambda}_0}{32\pi^2}\left(-\frac{1}{\varepsilon} + \ln\left(\frac{m^2}{\mu^2}\right) + f(p_1,p_2,p_3)\right)\right]$$

$$= -\tilde{\lambda}(\mu)\left[1 + \frac{3\tilde{\lambda}(\mu)}{32\pi^2}\left[\ln\left(\frac{m^2}{\mu^2}\right) + f(p_1,p_2,p_3)\right]\right]$$

where $\tilde{\lambda}(\mu)$ is the $\overline{MS}$ renormalized coupling. (For simplicity we from now on drop the bar on $\bar{\mu}$.)

$$\left[\, f(p_1,p_2,p_3) = V(s) + V(t) + V(u)\,; \quad \begin{aligned} s &= (p_1+p_2)^2 \\ t &= (p_1-p_3)^2 \\ u &= (p_1-p_4)^2 \end{aligned} \,\right]$$

To get reasonable higher order corrections, we need $\mu \approx m$.

So we're in trouble:

* matching requires $\mu \approx M$
* EFT matrix elements require $\mu \approx m$
* and $m \ll M$!

This problem would manifest itself in the full theory as terms of the form $\lambda^n \ln\left(\frac{m^2}{M^2}\right)$ and results in a breakdown of perturbation theory (in the $\overline{MS}$ scheme) for $m \ll M$, even if λ is very small.

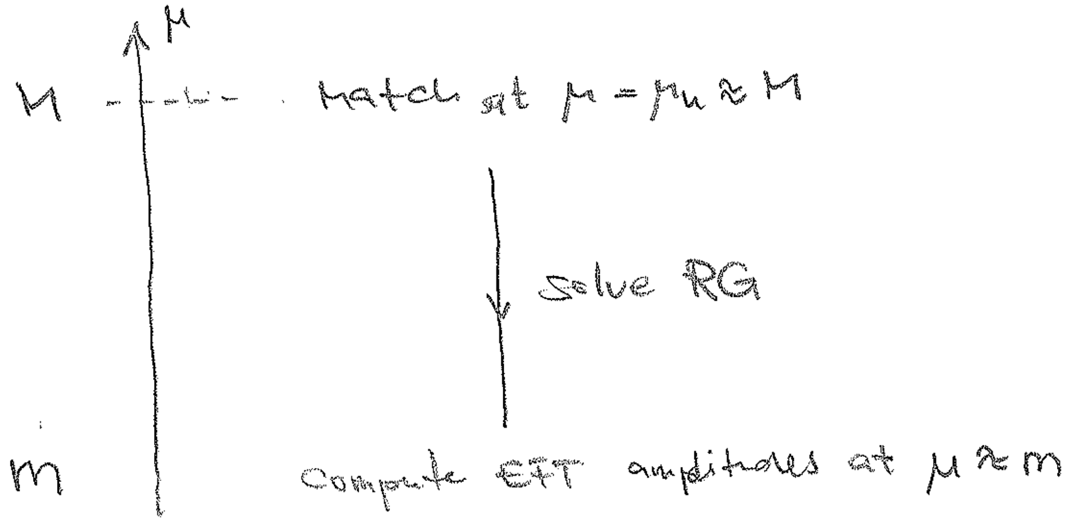
Fortunately, the renormalization group (RG) in the EFT allows us to resum the logarithmically enhanced terms to all orders by solving the RG evolution equations

$$\frac{d\tilde{\lambda}(\mu)}{d\ln\mu} = \mu \frac{d\tilde{\lambda}(\mu)}{d\mu} = \beta(\lambda(\mu))$$

and
$$\frac{d\,C_i(\mu)}{d\ln\mu} = \gamma_{ij}(\lambda(\mu))\,C_j(\mu)$$

this second equation is a matrix equation. Operators of the same dimension "mix".

The general strategy is then as follows:



Let us illustrate this for the leading-power four-point function. To get the β-function, use

$$\frac{d}{d\ln\mu}\mathcal{A} = 0 \qquad \left(\begin{array}{c}\text{physical amplitude is} \\ \mu\text{-indep}\end{array}\right)$$

$$\Longrightarrow \qquad \frac{d}{d\ln\mu}\tilde{\lambda}(\mu) = \beta(\lambda) = \frac{3\,\tilde{\lambda}^2(\mu)}{16\pi^2} + \ldots$$

Now solve the RG for the coupling constant:

$$\int_{\lambda(\mu_0)}^{\lambda(\mu)} \frac{d\tilde{\lambda}}{\tilde{\lambda}^2} = \frac{3}{16\pi^2} \int_{\ln(\mu_0)}^{\ln(\mu)} d \ln \mu$$

$$\rightarrow \quad \frac{1}{\tilde{\lambda}(\mu_0)} - \frac{1}{\tilde{\lambda}(\mu)} = \frac{3}{16\pi^2} \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\rightarrow \quad \tilde{\lambda}(\mu) = \frac{\tilde{\lambda}(\mu_h)}{1 - \frac{3}{16\pi^2} \tilde{\lambda}(\mu_h) \underbrace{\ln\left(\frac{\mu}{\mu_h}\right)}_{O(1)}} \qquad (*)$$

This solves the problem: use $\mu_h \cong M$

$$\rightarrow \quad \lambda(\mu_h) = \lambda(\mu_h) + \frac{3\lambda_{HL}^2}{32\pi^2} \Big[\ln\left(\frac{M^2}{\mu_h^2}\right) + c \Big]$$

small!

Then use $(*)$ to obtain $\lambda(\mu \approx m)$

Our method of performing computations is called Renormalization-Group (RG) improved perturbation theory. Instead of expanding in $\tilde{\lambda}(\mu)$, we expand in $\tilde{\lambda}(\mu_u)$ and $\tilde{\lambda}(\mu)$, which are both considered small, while logs $\ln\left(\frac{\mu}{\mu_u}\right)$ are counted as $O\left(\frac{1}{\lambda}\right)$.

(*) is accurate up to corrections of $O(\tilde{\lambda})$, but for our one-loop computation, we need to include these corrections to do so, one has to solve the two-loop RG

$$\mu \frac{d\tilde{\lambda}}{d\mu} = \beta(\tilde{\lambda}) = \tilde{\lambda}\left[3\frac{\tilde{\lambda}}{16\pi^2} - \frac{17}{3}\left(\frac{\tilde{\lambda}}{16\pi^2}\right)^2 + \dots\right]$$

see e.g. hep-ph/9503230

$$\rightsquigarrow \frac{3}{16\pi^2} \ln\left(\frac{\mu}{\mu_h}\right) = \frac{1}{\lambda(\mu_h)} - \frac{1}{\lambda(\mu)} + \frac{17}{9} \frac{1}{16\pi^2} \ln\left(\frac{\lambda(\mu)}{\lambda(\mu_h)}\right)$$

$$+ O(\lambda)$$

The fact that we need the β-functions and anomalous dimensions one order higher than the matching is a general feature of RG-improved perturbation theory.

While we were working in a toy model, the same problems arise whenever scale hierarchies are present. To get reliable results, one has to use an EFT and work in RG improved PT. With all the technology in place, we are now ready for real-life applications.

To finish our discussion, let us repeat the steps needed to construct the effective theory:

1.) Identify the degrees of freedom at low E

2.) Construct the most general $\mathcal{L}$ with these degrees of freedom and the symmetries of the full theory.

2a.) Higher-dim operators in $\mathcal{L}_{eff}$ are suppressed by $\left(\frac{1}{M}\right)^n$, where M is a characteristic high-energy scale. Their contribution to observables is suppressed by $\left(\frac{E}{M}\right)^n$, so only a finite number of terms is needed for an accuracy ε: $n \approx \ln(\varepsilon)/\ln(E/M)$.

2b.) Field redefinitions: higher-power terms in $\mathcal{L}_{eff}$ which vanish by the leading-power EOM do not contribute to physical amplitudes and can be omitted from $\mathcal{L}_{eff}$.

3.) Matching. If possible, determine the Wilson coefficients of the operators in $\mathcal{L}_{eff}$ by computing a number of quantities in both the full and the effective theory. Adjust the couplings in $\mathcal{L}_{eff}$ to reproduce the full theory result.

If 2b.) has been used, only physical quantities match, otherwise arbitrary Green's functions.

4.) RG improvement. Compute the anomalous dimensions and solve the RG equations for the operators in $\mathcal{L}_{eff}$. Solve these equations:

$$C_i(\mu) = \sum_j \underbrace{U_{ij}(\mu_h, \mu)}_{\substack{\text{Evolution} \\ \text{from } \mu_h \approx M \\ \text{to } \mu \approx E \\ \text{resums log's}}} \underbrace{C_j(\mu_h)}_{\substack{\text{No large log's} \\ \text{for } \mu_h \approx M}}$$