3.2. Field redefinitions

With our matching computation we ensured that our effective theory reproduces the full theory result for the off-shell Green's functions.

However, if we are only interested in physical quantities, such as scattering amplitudes, we can simplify the Lagrangian using field redefinitions.

As an example, consider

$$\phi_L \to \left[1 + \frac{\alpha}{M^2} \Box \right] \phi_L \, .$$

Let's plug this into $\mathcal{L}_{eff}$. Neglecting $\frac{1}{M^4}$ terms, we get

$$\mathcal{L}_{eff} \to \mathcal{L}_{eff} - \frac{\alpha}{M^2} \phi_L \Box \left(\Box + m^2 + \frac{\tilde{\lambda}}{3!} \phi_L^2 \right) \phi_L$$

By choosing $\alpha = -\frac{1}{2} C_{(2,4)}$, we cancel the term $-\frac{1}{2}\frac{C_{(2,4)}}{M^2}\phi\Box^2\phi$ in $\mathcal{L}_{eff}$ such that $\mathcal{L}'_{eff}$ no longer contains this term:

$$\mathcal{L}_{eff} \longrightarrow \mathcal{L}_{eff}\Big|_{C_{(2,4)}\to 0} + \frac{C_{(2,4)}}{2M^2}\phi_c\Box(m^2+\frac{\lambda}{3!}\phi^2)\phi_c$$

Note that the effect of the field redefinition can be obtained by using the leading-power equation of motion (EOM)

$$(\Box + m^2 + \frac{\lambda}{3!}\phi_c^2)\phi_c = 0$$

to eliminate higher-power terms in the Lagrangian.

Using redefinitions

$$\phi \longrightarrow \phi + \left(\frac{1}{M^2}\right)^n f(\phi) = \phi + \delta\phi$$

generates

$$\mathcal{L}_{eff} \longrightarrow \mathcal{L}_{eff} + \overbrace{\left(\frac{1}{M^2}\right)^n f(\phi)}^{\delta\phi}\overbrace{\left[\Box\phi + m^2\phi + \frac{\lambda}{3!}\phi^3\right]}^{\text{EOM, from } \frac{\delta S}{\delta\phi}}$$
$$+ O\left(\left(\frac{1}{M^2}\right)^{n+1}\right)$$

and allows one to systematically eliminate equation of motion terms from $\mathcal{L}_{eff}$.

It turns out that these field redefinitions leave the physics unchanged. This is true because:

1.) ϕ and ϕ' have the same quantum numbers, so after inserting states

$$\lim_{x^0 \to \infty} \langle 0 | T \{ \phi(x_1) \cdots \} | 0 \rangle$$

$$= \sum_X \langle 0 | \phi(x_1) | X \rangle \langle X | T \{ \cdots \} | 0 \rangle$$

the same amplitudes can be extracted from the theory, only the Z-factors $\langle 0 | \phi(0) | X \rangle = Z^{\frac{1}{2}}$ change.

2.) The Jacobian $\det\left(\frac{\delta\phi}{\delta\phi'}\right)$ is trivial, at least in dim. reg.

Let us illustrate the first point using an example. The tree-level $2 \to 2$ scattering amplitude in ϕ^4 is

$\mathcal{A} =$  $= -\lambda$

Let us now calculate this amplitude after the field redefinition $\phi \to \left(1 + \frac{\alpha}{M^2}\Box\right)\phi$.

$-p^2\,(-p^2+m^2)$

$$\mathcal{L}' = \mathcal{L} - \frac{\alpha}{M^2}\,\phi\,\Box\left(\Box + m^2 + \frac{\lambda}{3!}\phi^2\right)\phi$$

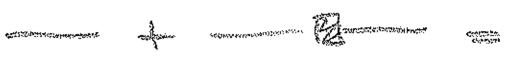
$\mathcal{A} =$  $+$  $+ \cdots +$ 

external leg correction, removed by amputation

$$= -\lambda\left(1 - \frac{\alpha}{M^2}\frac{4!}{3!}\cdot\frac{1}{4}\underbrace{\left(\sum_{i=1}^{4} p_i^2\right)}_{4m^2}\right)\left(Z^{+\frac{1}{2}}\right)^4$$

$$= -\lambda\left(1 - \frac{\alpha}{M^2}4m^2\right)\cdot\left(Z^{+\frac{1}{2}}\right)^4$$

To get the Z-factor, we need to evaluate the two-point function:



$$= \frac{i}{p^2-m^2} + \frac{2\alpha}{M^2}\frac{i}{p^2-m^2}(-ip^2)(p^2-m^2)\frac{i}{p^2-m^2}$$

$$= \frac{i}{p^2-m^2}\left(1+\frac{2\alpha}{M^2}p^2\right) = \frac{i\left(1+\frac{2\alpha}{M^2}m^2\right)}{p^2-m^2} + \text{"non-pole"}$$

$$\to \quad Z = \left(1+\frac{2\alpha}{M^2}m^2\right)$$

$$A = -\lambda\left(1-\frac{\alpha}{M^2}4m^2\right)\Big(\overbrace{1+\frac{\alpha}{M^2}m^2}^{Z^{1/2}}\Big)^4 = -\lambda + O\left(\frac{1}{M^4}\right)$$

So, we indeed get the same amplitude.

Finally, let us show why the Jacobian is trivial:

$$\int \mathcal{D}\phi = \int \mathcal{D}\phi' \underbrace{\det\left(\frac{\delta\phi}{\delta\phi'}\right)}_{J}$$

In our case $\phi \to \phi' + \left(\frac{1}{M^2}\right)^n f(\phi')$

$$\frac{\delta\phi(x)}{\delta\phi'(x')} = \delta(x-x') + \left(\frac{1}{M^2}\right)^n f'(\phi'(x))\,\delta(x-x')$$

The Jacobian can be written as

Grassmann fields

$$\det\left(\frac{\delta\phi}{\delta\phi'}\right) = \int \mathcal{D}c \int \mathcal{D}\bar{c} \; \exp\left[i \int d^dx \int d^dy \; \bar{c}(x) \frac{\delta\phi(x)}{\delta\phi'(y)} c(y) \right]$$

$$= \int \mathcal{D}c \int \mathcal{D}\bar{c} \; \exp\left[i \int d^dx \; \bar{c}(x) \left[1 + \left(\frac{1}{M^2}\right)^n f'(\phi) \right] c(x) \right]$$

Since the f' - term is $\frac{1}{M^2}$ suppressed, it should be treated as a perturbation. So the ghost diagrams are loops of a "fermion" with propagator $\frac{i}{1}$.

Such loops $\int d^dk \left(\frac{i}{1}\right)^n = 0$ in dim. reg.,

so $\det\left(\frac{\delta\phi}{\delta\phi'}\right) = 1$.