2.1. Renormalization group (à la Wilson)

So far, we have considered a situation, where we integrated out physics above some characteristic scale M. It is also interesting to look at what happens if we only integrate out a small slice $\Lambda > \omega > \Lambda - \delta\Lambda$ in which the particle content remains unchanged. In this case the form of the action is unchanged, only the coefficients g_i change. Repeating the procedure, one obtains the couplings as a function of the cutoff

$$\{g_i(\Lambda)\} \Rightarrow \{g_i(\Lambda - \delta\Lambda)\} \Rightarrow \{g_i(\Lambda - 2\delta\Lambda)\}$$

$$\Rightarrow \ldots$$

The evolution equations $\Lambda \frac{dg_i}{d\Lambda} = f(\{g_j\})$ are called Renormalization Group (RG) equations.

Let us derive this RG evolution for the trivial but instructive case of a quadratic action. The most general form is

$$\mathcal{L} = \frac{1}{2}\phi(x)\left[-m^2 - 1\times\Box + c\,\Box^2 + \ldots\right]\phi(x)$$

$\uparrow$ by convention (rescale field)

Let's Fourier transform

$$\phi(x) = \int_k e^{-ikx}\,\tilde{\phi}(k) = \int \frac{d^dk}{(2\pi)^d}\, e^{-ikx}\,\tilde{\phi}(k)$$

$$S = \frac{1}{2}\int d^dx \int_p\int_k \tilde{\phi}(p)\left[-m^2 + k^2 + ck^4 + \ldots\right]\tilde{\phi}(k)\, e^{-i(p+k)x}$$

$$= \frac{1}{2}\int_k \tilde{\phi}(-k)\left[-m^2 + k^2 + ck^4 + \ldots\right]\tilde{\phi}(k)$$

We assume that our theory is defined with an UV cutoff Λ:

$$\int_k \longrightarrow \int_k^{\Lambda} = \int_{-\Lambda}^{\Lambda}\frac{dk^0}{2\pi}\int_{-\Lambda}^{\Lambda}\frac{dk^1}{2\pi}\cdots\int_{-\Lambda}^{\Lambda}\frac{dk^{d-1}}{2\pi}$$

Let us now split $\phi = \phi_L + \phi_H$

$$\tilde{\phi}(k) = \tilde{\phi}_L(k) + \tilde{\phi}_H(k)$$

$$= \begin{cases} \tilde{\phi}_L(k) & |k_\mu| < b\Lambda \quad \forall \mu \\ \tilde{\phi}_H(k) & |k_\mu| > b\Lambda \text{ for some } \mu \end{cases}$$

The field ϕ_L describes fluctuations below $\Lambda' = b\Lambda$.
Our action splits

$$S = S_L + S_H = \frac{1}{2} \int_k^{b\Lambda} \tilde{\phi}_L(k) [\ldots] \tilde{\phi}_L(k)$$
$$+ \frac{1}{2} \int_k^{\Lambda} \tilde{\phi}_H(k) [\ldots] \tilde{\phi}_H(k)$$

If we are only interested in low energy Green's functions

$$\langle 0 | T \{ \phi_L(x_1) \ldots \phi_L(x_n) \} | 0 \rangle = \frac{1}{Z} \int \mathcal{D}\phi_L \int \mathcal{D}\phi_H$$
$$* \, e^{iS_H} e^{iS_L} \phi_L(x_1) \ldots \phi_L(x_n)$$

then the effect of ϕ_H is absorbed by the normalization, so

$$\langle 0 | T \{ \phi_L \ldots \phi_L \} | 0 \rangle = \frac{1}{Z_L} \int \mathcal{D}\phi_L \, e^{iS_L} \phi_L(x_1) \ldots \phi_L(x_n)$$

To compare S_L with S, let us rescale

$$k' = \frac{k}{b} \quad ; \quad x' = x\,b$$

In terms of the variable k', the cut-off is back at Λ. The action becomes

$$S_L = \int_{k'}^{\Lambda} b^d \, \tilde{\phi}(k') \left[m^2 + b^2 k'^2 + b^4 c\, k'^4 + \dots \right] \tilde{\phi}(k')$$

Let us further rescale $\tilde{\phi} \to \tilde{\phi}' \cdot b^{-\frac{d+2}{2}}$ to have a canonically normalized kinetic term.

$$S_L = \int_{k'}^{\Lambda} \tilde{\phi}' \left[\frac{m^2}{b^2} + k'^2 + b^2 c + \dots \right] \tilde{\phi}$$

So we get the same theory, but with

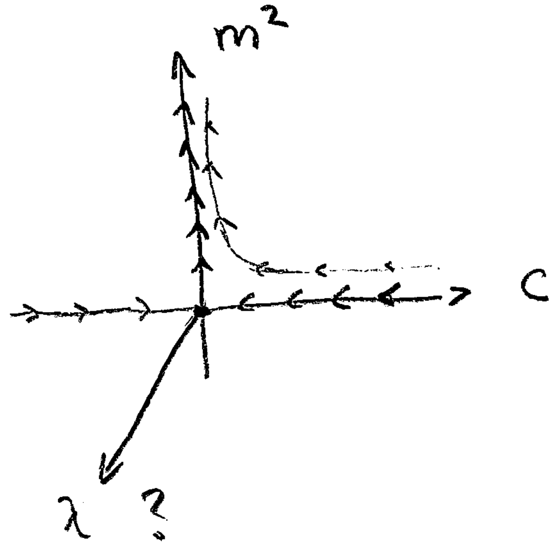
$$m^2 \to \frac{m^2}{b^2} \qquad \text{"relevant"}$$

$$c \to b^2 c \qquad \text{"irrelevant"}$$

...

So for $b = \frac{1}{2}$, for example, the mass becomes twice as large, while the coefficient of the four-derivative term is four times smaller.

If we iterate the transformation, we get the "renormalization group flow" in the space of coupling constants:



The point $m = c = \cdots = 0$, the massless scalar field action, is a <u>fixed point</u>. This is called the "Gaussian fixed point".

When this analysis is extended to theories which include <u>small</u> couplings, the result is basically unchanged. The irrelevant operators remain irrelevant, and the relevant ones stay relevant.

However, it becomes very interesting to check what happens with marginal operators. The small perturbation induced by the couplings will make them marginally relevant, or marginally irrelevant.

For QCD, it turns out that the coupling slowly gets stronger as the high energy modes are integrated out. Starting with an essentially free theory defined with a very high cut-off Λ, one ends up with a strongly coupled theory at low energy. This property is called "asymptotic freedom".

As we will show now, the situation is opposite for ϕ^4 theory. Even if the theory has a large coupling in it's Lagrangian, it looks more and more like a free theory when the high-energy modes are integrated out.

So let's look at ϕ^4 theory with a cut-off Λ

$$Z = \int \mathcal{D}\phi \exp\left[- \int d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right) \right]$$

$$\mathcal{D}\phi = \prod_{|k| < \Lambda} d\tilde{\phi}(k)$$

I've switched to Euclidean space $t_M = -i t_E$, $x_M^2 = -x_E^2$, in the above expression $x^\mu \equiv x_E^\mu$, $k \equiv k_E^\mu$.

In Minkowski space

$$S = \int d^d x_M \left[\frac{1}{2} (\partial_\mu^M \phi)^2 - m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right]$$

$$= -\frac{1}{i} \int d^d x_E \left[\frac{1}{2} (\partial_\mu^E \phi)^2 + m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

$$= -\frac{1}{i} S_E$$

Let us now again split

$$\tilde{\phi}(k) = \tilde{\phi}_L(k) + \tilde{\phi}_H(k)$$

$$\tilde{\phi}_H(k) = \tilde{\phi}(k) \, \Xi(k)$$

$$\Xi(k) = \Theta(|k| < \Lambda) \, \Theta(|k| > b\Lambda)$$

The quadratic part of the action will again just turn into a sum of quadratic actions, but the interaction now also includes cross terms:

$$S(\phi_L + \phi_H) = S(\phi_L) + S(\phi_H)$$
$$+ \int d^4x\, \lambda \left[\frac{\phi_L \phi_H^3}{3!} + \frac{\phi_L^2 \phi_H^2}{2!2!} + \frac{\phi_L^3 \phi_H}{3!} \right]$$

Now we'll derive Feynman rules and then integrate over ϕ_H to lowest order in PT.

Propagators:

$$\Delta_L = \langle 0 | T\{ \phi_L(x)\, \phi_L(0) \} | 0 \rangle = \int^{\Lambda} \frac{d^4k}{(2\pi)^4}\, e^{ikx} \frac{1}{k^2+m^2}$$

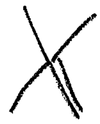
$$\Delta_H = \langle 0 | T\{ \phi_H(x)\, \phi_H(x) \} | 0 \rangle = \int^{\Lambda} \frac{d^4k}{(2\pi)^4}\, e^{ikx} \frac{1}{k^2+m^2}\, \Theta(|k| > b\Lambda)$$

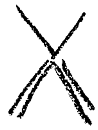
$$\langle 0 | T\{ \phi_H(x)\, \phi_L(x) \} | 0 \rangle = 0$$

Let us denote Δ_L = ————

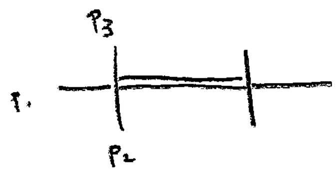
Δ_H = ════

The Feynman rules for the interactions are

 $= i\lambda$  $= i\lambda$

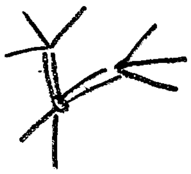
 $= i\lambda$  $= i\lambda$

At the tree-level, we have diagrams such as

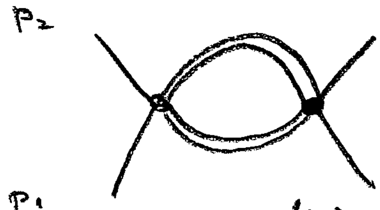


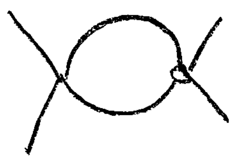
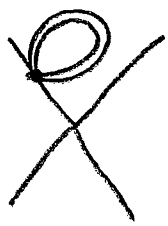
$$= \frac{1}{(p_1+p_2+p_3)^2+m^2}\,\Theta\left(|p_1+p_2+p_3| > \Lambda b\right)$$

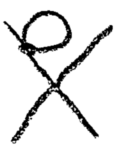
which corresponds to a $\frac{1}{(\Lambda b)^2}$ suppressed ϕ_L^6 interaction after integrating out ϕ_H.

 $\sim \frac{1}{(\Lambda b)^4}\,\phi_L^8$ interaction, etc.

We are interested in the behavior of the ϕ^4 interaction. In this case, there is no tree-level contribution, but one-loop diagrams of the form

(1) 

$+$ (2)  $+$ (3) 

$+$ (4)  $+$ perms

(2) + (4) are diagrams in the low-energy theory, so we only need to evaluate (1) and (3). It turns out that (3) only contributes to the mass term ϕ_L^2 but not to the interaction.

Finally, for (1) we get

$$D_1 = \underbrace{\frac{\lambda^2}{2}}_{\text{symmetry factor}} \int d^d k \, \frac{1}{k^2+m^2} \cdot \frac{1}{(k+p_1+p_2)^2+m^2}$$

$$\Theta(|k|>\Lambda b)\,\Theta(|k|<\Lambda)$$

$$\Theta(|k+p_1+p_2|>\Lambda b)\,\Theta(|k+p_1+p_2|<\Lambda)$$

because $m \ll \Lambda b$, $p_i^\mu \ll \Lambda b$, we can Taylor expand on the level of the integrand. Higher orders in the expansion are suppressed by $\frac{m}{M}$, $\frac{p^\mu}{M}$ and match onto irrelevant operators.

$$D_1 = \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2} \Theta(|k|>\Lambda b)\,\Theta(|k|<\Lambda)$$

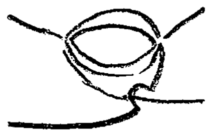
$$+ \dots$$

$$= \frac{\lambda^2}{2} \frac{\Omega_d}{(2\pi)^d} \int_{b\Lambda}^{\Lambda} dk\, k^{d-5} = \frac{\lambda^2}{2}\Omega_d \frac{\overbrace{\Lambda^{d-4}-(\Lambda b)^{d-4}}^{-\ln(b)}}{d-4}$$

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \, . \qquad \Omega_4 = 2\pi^2$$

So for $d=4$, we have

$$D_1 = \frac{\lambda^2}{16\pi^2} \ln\left(\frac{1}{b}\right)$$

Accounting also for  and 

yields the result

 $$= \frac{3\lambda^2}{16\pi^2} \ln\left(\frac{1}{b}\right)$$

In the low energy theory, this contribution must arise from $-\int d^4x \, \frac{\lambda'}{4!} \phi_L^4$, so we must have

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \ln\left(\frac{1}{b}\right)$$

The coupling gets <u>weaker</u> when high energy modes are integrated out!

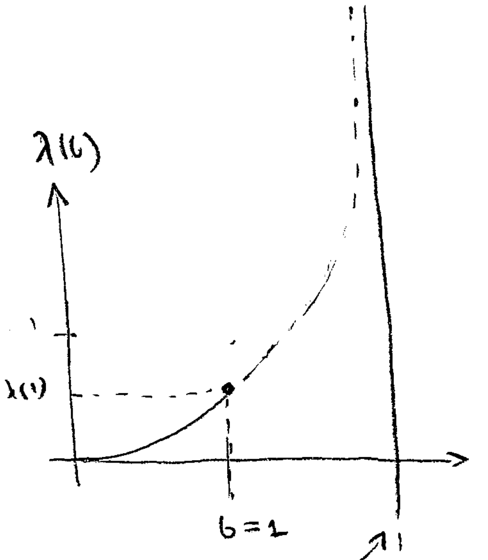
Let us imagine that we integrate the high-energy physics little by little, so

$$d\lambda = + \frac{3\lambda^2}{16\pi^2}\, d\ln b$$

$$\Rightarrow \int_{\lambda(1)}^{\lambda(b)} \frac{d\lambda}{\lambda} = \frac{3}{16\pi^2}\left[\ln(b) - \ln(1)\right]$$

$$\Rightarrow \frac{1}{\lambda(b)} - \frac{1}{\lambda(1)} = \frac{3}{16\pi^2}\ln(b)$$

$$\Rightarrow \lambda(b) = \frac{\lambda(1)}{1 + \frac{3}{16\pi^2}\lambda(1)\ln\left(\frac{1}{b}\right)}$$



Landau pole: $-1 = \frac{3}{16\pi^2}\lambda(1)\ln\left(\frac{1}{b}\right)$

Our analysis is a bit simplistic, in that we only look at one operator, ϕ^4, and do not include the effects of "irrelevant" operators. Also, our conclusion is only valid at small coupling.

Nevertheless, all available (perturbative and non-perturbative) evidence suggests that the behavior persists at arbitrary values of λ.

Since the coupling becomes small, one needs to start out with a sufficiently strong coupling at large value of the cut-off

$$\lambda(1) = \frac{\lambda(b)}{1 - \frac{3}{16\pi^2}\lambda(b)\ln(\frac{1}{b})}$$

Since $\lambda(1) \to \infty$ for $\ln(\frac{1}{b}) = \frac{16\pi^2}{3}\frac{1}{\lambda(b)}$

it appears that the cutoff cannot be arbitrarily large, $\ln(\frac{1}{b}) = \ln\left(\frac{\Lambda}{E}\right) \leq \frac{16\pi^2}{3}\frac{1}{\lambda(1)}$

Since we did our analysis in perturbation theory, but are looking at $\lambda \to \infty$ it is obviously not very meaningful. Theories which have the property that the cut-off cannot be chosen arbitrarily large for non-vanishing λ are called <u>trivial</u>.

All evidence strongly suggests that $\lambda\phi^4$ is a trivial theory.