2. The Wilsonian effective action

Consider a field theory with a characteristic large energy scale M, and suppose we are only interested at physics at low energies $E \ll M$.

This is the physics situation effective theories are designed to analyze. The full theory is defined in terms of a path integral. Everything we wish to know can be obtained from calculating the expectation values

$$\langle 0 | T\{ \phi(x_1) \dots \phi(x_n) \} | 0 \rangle$$

$$= \frac{1}{Z} \int \mathcal{D}\phi \, e^{iS(\phi)} \, \phi(x_1) \dots \phi(x_n)$$

$$\prod_{x^\mu} \int d\phi(x) \quad \text{or} \quad \prod_{p^\mu} \int d\tilde{\phi}(p) \quad ; \quad Z = \int \mathcal{D}\phi \, e^{iS(\phi)} .$$

To obtain the low-energy effective action, split the field

$$\phi = \phi_L + \phi_H$$

where ϕ_H contains all Fourier modes with $\omega \geq \Lambda$ and ϕ_L the low energy modes $\omega < \Lambda$. Since we are only interested in low energy physics, we only want correlation functions

$$\langle 0 | T\{ \phi_L(x_1) \ldots \phi_L(x_n) \} | 0 \rangle$$

$$= \int d\phi_L \underbrace{\int d\phi_H \; e^{iS(\phi_L + \phi_H)}} \phi_L(x_1) \ldots \phi_L(x_n)$$

$$= \int d\phi_L \; e^{iS_\Lambda(\phi_L)} \phi_L(x_1) \ldots \phi_L(x_n) \, .$$

$S_\Lambda(\phi_L)$ is called the "Wilsonian effective action" and we have chosen $\Lambda \lesssim M$ to integrate out the physics associated with M.

$S_\Lambda(\varphi_L)$ is non-local on scales $\Delta x^\mu \sim \frac{1}{\Lambda}$, because high-energy fluctuations have been integrated out.

As a final step one expands the nonlocal action as a series of local operators. This expansion is possible because $E \ll \Lambda$. The result has the form

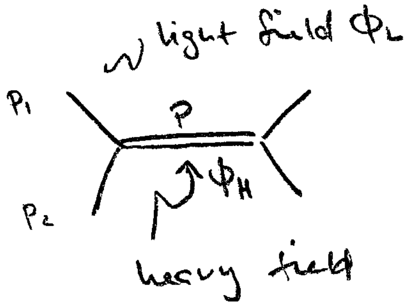
$$S_\Lambda(\phi_L) = \int d^dx \, \mathcal{L}_\Lambda^{eff}(x)$$

$$\mathcal{L}_\Lambda^{eff}(x) = \sum g_i \, O_i(x).$$

The object $\mathcal{L}_\Lambda^{eff}$ is called the "effective lagrangian". It is an infinite sum over local operators O_i allowed by symmetries. The coefficients g_i are referred to as Wilson coefficients.

To make this a little more concrete, assume that we integrated out a heavy particle with mass M.

The full theory might contain diagrams



$$\sim \frac{1}{p^2 - M^2}$$

Since $p_1, p_2 \ll M$, we can expand

$$\frac{1}{p^2 - M^2} = -\frac{1}{M^2} + \frac{p^2}{M^4} \hat{=} -\frac{1}{M^2}\,\delta^{(4)}(x) - \frac{\Box}{M^4}\,\delta^{(4)}(x)$$

so $\mathcal{L}_{eff}$ will contain terms such as $\phi_L^4(x)$ and $\partial_\mu\phi\,\partial^\mu\phi\,\phi^2(x)$, etc.

In general it will be very hard to calculate the coefficients g_i, and since we ended up with infinitely many terms in $\mathcal{L}_{eff}$ it is unclear how the construction is useful.

What saves the day is dimensional analysis.

With $\hbar = c = 1$ $[m] = [E] = [x^{-1}] = [t^{-1}]$ are all measured in the same units. Assume that $[g_i] = -\gamma_i$ is the mass dimension of g_i. It then follows that

$$g_i = c_i M^{-\gamma_i}$$

with a dimensionless coefficient c_i. Since the coefficients g_i arose when integrating out the physics associated with M, it is natural to assume that $c_i \sim 1$. Very large, e.g. $c_i \sim 10^6$, or very small coefficients, e.g. $c_i \sim 10^{-6}$, would call for some explanation.

At low energy, the contribution of O_i to a dimensionless observable scales as

$$\left(\frac{E}{M}\right)^{\gamma_i} = \begin{cases} O(1) & \gamma_i = 0 \\ \gg 1 & \gamma_i < 0 \\ \ll 1 & \gamma_i < 0 \end{cases}$$

It follows that only operators with $\gamma_i \leq 0$ are important at low energy.

To derive the mass dimension δ_i of a given operator, we need to know the mass dimension of the fields. Assume that the theory is weakly coupled. Then the scaling dimension is determined by the free action

$$S_0 = \int d^D x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 \right)$$

with $x \sim \frac{1}{E}$, $\partial_\mu \sim E$ and using that the action is dimensionless, we find

$$\phi \sim E^{D/2 - 1}.$$

For an operator with mass dimension δ_i, we have $\gamma_i = \delta_i - D$.

Let's now look at some examples

	δ_i	γ_i	g_i
$\partial_\mu\phi\,\partial^\mu\phi$	D	0	1
ϕ^2	$D-2$	-2	$\sim M^2$
ϕ^4	$2D-4$	$D-4$	M^{4-D}
$(\partial_\mu\phi)^2\phi^2$	$2D-2$	$D-2$	M^{2-D}
ϕ^6	$3D-6$	$2D-6$	M^{6-2D}

For an operator with n fields and m derivatives, we have

$$\delta_i = n\left(\frac{D}{2}-1\right)+m \; ; \quad \gamma_i = (n-2)\left(\frac{D}{2}-1\right) + (m-2)$$

so only very few operators have $\gamma_i \leq 0$ (unless $d \leq 2$!)

The following terminology is commonly used:

Dim.	Importance for $E \to 0$ / $M \to \infty$	Terminology for operator
$\delta_i < D$ $\gamma_i < 0$	grows	relevant (super-renormalizable)
$\delta_i = D$ $\gamma_i = 0$	constant	marginal (renormalizable)
$\delta_i > D$ $\gamma_i > 0$	declines	irrelevant (non-renormalizable)

The terminology is not optimal. For example, it is interesting to search for the effects mediated by irrelevant operators, since they provide information on the physics at very high energies.

Our discussion makes it clear that renormalizability is overrated: usually one avoids irrelevant operators because they render a theory nonrenormalizable. However, once we admit that a theory is not valid up to infinitely large energies, then it is clear that it will contain also irrelevant operators. This is not a problem, because their contributions are suppressed by some large scale M, at which new physics enters.

Renormalizable Lagrangians are so successful in describing our low-energy measurements, because the relevant and marginal operators are the most important at low energies.

Examples of irrelevant operators:

- For example, the gauge symmetries of the standard model allow us to write down a dimension 5 term $g\mathcal{O} = g\,\nu^T H H \nu$ with $g \sim \frac{1}{\Lambda}$, $\nu \sim E^{3/2}$

After EW symmetry breaking, this yields a Majorana mass term for the neutrino, with $m_\nu \sim \frac{\langle H \rangle^2}{\Lambda}$.

The fact that $\Lambda \sim 10^{14}\,\text{GeV}$ is seen as indication for a Grand Unified Theory.

– The weak interaction is so weak at low energies because mediated by irrelevant operators such as

$$\mathcal{O} = \bar{u}\gamma_\mu(1-\gamma_5)d \; \bar{e}\gamma^\mu(1-\gamma_5)\nu$$

The fermion field $\psi \sim E^{3/2}$, so $\delta = 6$, $\gamma = 2$.

$\to$ The coefficient of the operator must $\sim \frac{1}{M^2}$.

Here, the mass $M = M_W$, the mass of the W-boson.

From the form of the interaction, Oskar Klein predicted the existence of massive particles with $M_W \gtrsim 60$ GeV already in 1938.

While irrelevant operators are perfectly natural, super-renormalizable / relevant operators are problematic. Consider for example the ϕ^2 operator in ϕ^4 theory. In $D=4$, we have $\delta_i = 2$, $\gamma_i = -2$, and so we expect that $m^2 \sim \Lambda^2$! Integrating out quantum fluctuations at large scales generates a large mass for scalar particles.

But this is a contradiction: if $m^2 \sim \Lambda^2$ we should have integrated out the corresponding field ϕ!
Note that also $\bar{\Psi}\Psi \sim E^3$ is relevant.

This reasoning leads one to conclude that only theories in which mass terms are forbidden by symmetries are natural.

Looking at the Standard Model as an effective theory, this condition is almost fulfilled:

- gauge bosons do not have mass terms because they are forbidden by gauge symmetry, e.g.

$$m^2 (A_\mu)^2 \xrightarrow{\text{gauge trans.}} m^2 e^{2i\alpha} (A_\mu)^2$$

is not invariant.

- fermion fields do not have mass terms because

$$\Psi_L = \tfrac{1}{2}(1-\gamma_5)\Psi$$
$$\Psi_R = \tfrac{1}{2}(1+\gamma_5)\Psi$$

have different gauge charges in the SM: Ψ_R is neutral under $SU(2)$, Ψ_L is not.

a mass term $m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$

would therefore violate gauge invariance.

Note that the absence of a mass term in $\mathcal{L}_{SM}$ does not imply that the fermions and gauge bosons are massless. They receive their mass by interacting with the Higgs condensate $\bar{\psi}_L H \psi_R \rightarrow \bar{\psi}_L \underbrace{\langle H \rangle}_{\text{vacuum expectation value}} \psi_R$.

The only mass term in the Standard Model is the mass term of the Higgs field $\mu^2 H^* H$.

There are several ways out of this dilemma, but all of them involve New Physics around the scale of the Higgs mass.

- Supersymmetry relates fermions and bosons. It can be used to protect scalar masses. Construct the theory such that fermion masses are forbidden $\rightarrow$ scalar masses are forbidden as well.

- In technicolor models, the Higgs is a bound state of a fermion - antifermion pair, similar to mesons in QCD.
- In little Higgs models, the Higgs is a pseudo Goldstone boson of a spontaneously broken global symmetry.

Alternatively, the smallness of m_H could just be due to some accidental cancellation. To make this more plausible, people often invoke the anthropic principle. " If the universe [in our example m_H] would be much different, we would not ~~be~~ here."