Effective Field Theory

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Abstract

This lecture provides an introduction to the framework of low-energy effective field theories. After developing the basic concepts, the method is used to analyze electromagnetic, weak, and strong interactions at low energies. The course is intended for master or graduate students who have taken a first course in quantum field theory.

[**The present script will be further extended and adapted as the course progresses. Please send suggestions and corrections to** `thomas.becher@unibe.ch`.]

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Preface

Effective field theory was first developed in the context of the strong interactions [1–3], but has since become an important tool in all of particle and nuclear physics (and beyond). It is based on ideas related to the Wilsonian renormalization group [4], which describes the evolution of operators as a function of the renormalization scale. In the first two chapters, we will review its basics and introduce the concepts and techniques of effective field theory using the example of scalar fields. The later chapters then address applications, mainly to the low-energy properties of the Standard Model.

These notes were compiled for the course "Effective Field Theory" by Thomas Becher first delivered at the University of Bern in 2010 and 2015, and set in LaTeX by Jonas Haldemann. The present version has been revised and corrected by Martin Hoferichter for a course in 2021. The material covered is largely based on Refs. [5–8], to which we also refer for further reading. Of course, we take full responsibility for all typos and mistakes introduced or overlooked by us.

1. Introduction

Low-energy effective quantum field theories – or effective field theories (EFTs), for short – are an important, widely used tool in particle physics and beyond. An *effective* theory, is a theory which only describes the physics below some energy scale Λ, as opposed to a *full* theory, which should be valid up to arbitrary high energies.

What are examples of effective theories? Every theory we know! For example, no one in his right mind believes that the Standard Model (SM) of particle physics is valid up to arbitrarily high energies. Quantum Chromodynamics (QCD), the theory of the strong interaction, by itself, would be consistent as a theory valid up to arbitrarily high energies since it has a special property called asymptotic freedom which makes it very well-behaved at asymptotically large energies. However, a detailed theoretical analysis of the rest of the SM strongly suggests that while it could be valid up to a very high scale Λ, much higher than what is currently accessible in experiments, it ceases to make sense as a theory beyond this energy. We will discuss this point later in the lecture. On top of this, there are also many physics reasons why the SM should be replaced by a different theory at high energies (a need for additional sources of CP violation, dark matter, quantum gravity, . . .). These arguments show that the SM must be viewed as an effective theory, even though we don't know at what scale Λ it breaks down.

Once one admits that a theory is only valid up to some scale Λ, it makes sense to perform an expansion in the ratio $\lambda = E/\Lambda$, where E is the physical energy scale one is interested in and Λ the scale at which the theory breaks down. Such expansions often greatly simplify the physics, similar to the expansion in small coupling constants which is the basis for perturbation theory, and enable many predictions and computations which would be impossible in the full theory, even if it is known. All computations of physical observables in the SM rely on effective field theory techniques, although this is not always explicitly spelled out in the literature.

Expansions in scale ratios

Expansions in scale ratios are also commonly used in classical physics. For example, we often deal with non-relativistic systems, where the three momentum (in the appropriate frame) is much smaller than the mass of a particle, $|\vec{p}| \ll m$.[1] In this case one can expand

$$E = \sqrt{m^2 + p^2} = m + \frac{\vec{p}^2}{2m} - \frac{(\vec{p}^2)^2}{8m^3} + \dots . \tag{1.1}$$

The expansion to second order is used in classical mechanics, but one can systematically add higher-order terms to account for relativistic corrections and achieve any desired precision, as long as $|\vec{p}| \ll m$. However, as $|\vec{p}| \sim m$, the expansion starts to break down. One can thus view non-relativistic mechanics as an effective theory of relativistic mechanics, which would be the full, fundamental theory. The same limit is also the basis for non-relativistic quantum mechanics. For example, in the hydrogen atom, $\langle \vec{p}^2 \rangle \sim (m\,\alpha)^2$, where $\alpha \sim 1/137$ is the fine structure constant and the higher order terms in (1.1) induce small, calculable corrections

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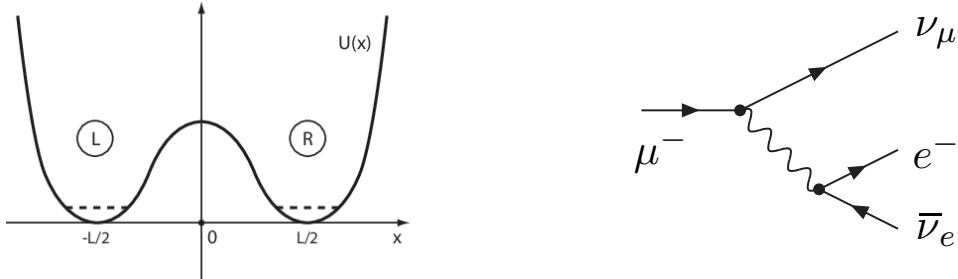[1] Throughout we use natural units $\hbar = c = 1$.

Figure 1.1.: Left: Double-well potential. Right: Weak decay of a muon. The wiggly line indicates the W-boson which mediates the decay.

to the spectrum. Indeed, quantum mechanics (QM) should be viewed as the effective theory of QED, which in turn is an effective theory of the SM. Hence, the computation of the hydrogen energy levels involves a tower of effective theories. One starts first with the SM, then transitions to QED, an effective theory that only includes light particles with masses $m \ll \Lambda \sim m_W, m_Z, \ldots$. In the second step, one takes the non-relativistic limit to end up with QM. The systematic construction of the non-relativistic theory is actually quite challenging. It is called non-relativistic QED (NRQED) and contains the quantum mechanics of the electron together with quantized photons, so it is still a quantum field theory.

As a second illustrative, classical example, consider the electromagnetic potential $\phi(\vec{r})$ of a static charge distribution $\rho(\vec{R})$ which is concentrated inside a small region $|\vec{R}| < R_0$. If we consider the potential for $r = |\vec{r}| \gg R_0$, we can expand in the ratio $\lambda \sim r \gg R_0$. This expansion takes the form

$$4\pi\epsilon_0\phi(\vec{r}) = \int d^3R \frac{\rho(\vec{R})}{|\vec{r} - \vec{R}|} = \frac{\rho_0}{r} + \frac{\rho_i\, r_i}{r^3} + \frac{\rho_{ij}\, r_i r_j}{2r^5} + \ldots \tag{1.2}$$

This is called the multi-pole expansion and the coefficients ρ_0, ρ_i, and ρ_{ij} are referred to as monopole-, dipole- and quadrupole-moments of the charge distribution $\rho(\vec{R})$. Since it is performed on the level of the integrand, we can view it as a distribution expansion

$$\rho(\vec{R}) = \rho_0\, \delta^{(3)}(\vec{R}) - \rho_i\, \partial_i\delta^{(3)}(\vec{R}) + \frac{1}{2!}\rho_{ij}\, \partial_i\partial_j\delta^{(3)}(\vec{R}) + \ldots \tag{1.3}$$

of the charge distribution, i.e. the systematic approximation of the charge distribution of a small object as a general local function (more precisely distribution). The hydrogen atom is again a good example of a case where this expansion is relevant. Even if we treat the proton as static (and classical), it has a very complicated charge distribution since it is a complicated bound state of quarks and gluons. However, the size of the proton is many orders of magnitude smaller than the size of a hydrogen atom and we can therefore treat it as a point source using (1.3).

Classical versus quantum theory

There is an interesting difference between classical and quantum mechanics when it comes to low-energy effective theories. In a classical theory, the states with high energy can never be reached when working at low energies. If there is a potential barrier with height $\Lambda \gg E$ classical particles will never cross it. As an example, if a classical particle is subjected to the potential shown on the left-hand side of Figure 1.1, it cannot cross the barrier if its energy is below it.

In a quantum theory, on the other hand, particles can tunnel through potential barriers, such as the one shown in Figure 1.1. Quantum particles are sensitive to the presence of high-energy states and this needs to be accounted for in the low-energy effective theory. One place where this becomes manifest is when doing perturbation theory for a Hamiltonian $\hat{H} = \hat{H}^0 + \lambda \hat{H}'$. Expanding the energy values as

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \cdots, \tag{1.4}$$

the second-order term gets contributions from unperturbed states with any energy E_m

$$E_n^1 = H'_{nn}, \qquad E_n^2 = \sum_{m \neq n} \frac{|H'_{nm}|^2}{E_n^0 - E_m^0}, \qquad H'_{nm} = \langle \psi_n^0 | \hat{H}' | \psi_m^0 \rangle. \tag{1.5}$$

A concrete example of this effect in quantum field theory is the weak decay of a muon $\mu^- \to \nu_\mu e^- \bar{\nu}_e$. This first occurs in second order in perturbation theory and the associated Feynman diagram is shown on the right-hand side of Figure 1.1. The mass of the muon is $m_\mu = 106\,\text{MeV}$, much lower than the mass of the W-boson $M_W = 80.4\,\text{GeV}$, so there is no actual W-boson produced in the decay. The W-boson is called a *virtual* particle. To account for the decay in the low energy theory, one has to introduce an explicit four-fermion interaction, see the right-hand side of Figure 1.2. In fact, Fermi theory was introduced in 1934 and is thus much older than the SM. To obtain the Fermi interaction from the SM result, one expands the propagator of the intermediate W-boson as

$$\frac{1}{q^2 - M_W^2} = -\frac{1}{M_W^2} \left(1 + \frac{q^2}{M_W^2} + \dots \right). \tag{1.6}$$

Since $q^2 = (p_\mu - p_{\nu_\mu})^2 \ll M_W^2$ the higher order terms in the expansion only have a very small effect. For precision calculations of the weak decay, one includes the first correction term.

The expansion (1.6) changes the high-energy behavior of the theory. While the original propagator falls off for $q^2 \to \infty$ this is no longer true for the expanded one. One computing higher-order terms in perturbation theory in the effective theory, the changed high-energy behavior leads to additional ultra-violet (UV) divergences not present in the full theory. These divergences arise because in expressions such as E_n^2 in (1.5), the sum (or integral for a continuum of states) runs up to infinitely large energies. To obtain meaningful results, one has to introduce an ultraviolet cutoff Λ_{UV} to regularize the effective theory.

Such UV divergences are always present in quantum field theories, but they arise already in ordinary quantum mechanics, if expansions such as (1.1) are (1.2) performed since the operators are singular at high energies or, equivalently, low energies. Of course, the physical results should be independent of Λ_{UV}. This can be achieved through a process called *renormalization*: In the presence of the cutoff, also the coupling constants of the theory will depend on Λ_{UV} and this dependence can cancel the one encountered the computations.
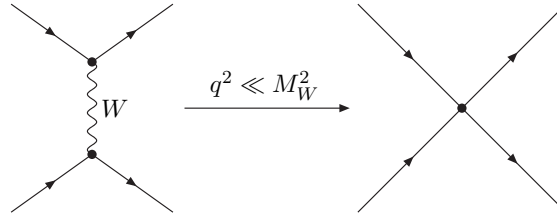
Figure 1.2.: Description of the weak decay in Fermi theory, which is also known as Weak Effective Theory (WET). Fermi theory includes an explicit four-fermion interaction to describe the decay (shown on the right side), while it proceeds through the exchange of a W-boson in the SM (left side).

EFTs provide a modern, physical understanding of renormalization. These theories are only valid up to a finite energy scale Λ and it therefore makes sense to define them with an ultraviolet cutoff $\Lambda_{UV} \simeq \Lambda$. At the same time, low-energy physics should not depend on the details of physics at arbitrarily high scales, so it should be possible to arrive at physical results that are independent of Λ_{UV} at the end of the day.

Wilsonian versus continuum effective theories

There are two ways of constructing effective theories. The first one is due to Wilson. In this approach, one obtains the low-energy theory by explicitly integrating out high-energy physics above a scale Λ in the path integral of the full theory. This approach is very useful to discuss many of the conceptual issues in the construction of effective theories. In particular, we will use it to classify operators in the effective theory. However, explicitly integrating out all physics above some scale is typically harder than solving the full path integral and it is therefore difficult to use this top-down approach in practice.

A much easier, bottom-up way is to construct the effective Lagrangian including all terms which are relevant at the necessary accuracy. One introduces the terms with arbitrary couplings and then determines the couplings with a *matching calculation*. One computes the same observable in the full and effective theory and then adjusts the couplings in the EFT to reproduce the full theory results to the desired accuracy. This approach is called continuum effective theory because one treats the EFT as a standard quantum field theory and typically works with dimensional regularization instead of a momentum or lattice cutoff.

In our lecture, we will first discuss the Wilsonian approach, but will then use continuum effective theory for all practical computations.

Why effective theories?

One may wonder why it is necessary or interesting to construct effective theories in cases where the full theory is known. For example, why use Fermi theory now that we have the SM? There are many reasons why effective theories are useful:

- **Simplification.** Expanding in scale ratios $\lambda \ll 1$ often drastically simplifies a theory. Computing Feynman diagrams in WET is much simpler than in the full theory (and there are typically fewer diagrams).

- **Factorization.** Effective theory allows one to separate the physics associated with different scales. This is especially useful in QCD, where the high-energy physics can be computed in perturbation theory, while the low-energy part typically requires a non-perturbative treatment.

- **Perturbation theory.** In problems with large scale hierarchies, higher-order terms in the coupling constant g are enhanced by logarithms of the scale ratio λ, i.e. the perturbative corrections involve terms $g^n \ln^m \lambda$, where normally $m \leq n$.[2] The large logarithms of λ can significantly enhance the higher-order corrections and overwhelm the suppression by the coupling, even at weak coupling. These logarithmic terms can be summed to all orders using the *renormalization group* in the EFT.

- **Approximate and emergent symmetries.** In some cases, the effective theory involves additional symmetries at leading power in λ. Examples are the so-called heavy quark symmetry and the chiral symmetry. Using EFTs, the consequences of these approximate symmetries can be systematically studied.

On top of this, effective field theory provides a systematic framework to parameterize the effects of unknown physics at higher energies, which is relevant for searches of physics beyond the SM.

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[2]In some more complex effective theories, such as Soft-Collinear Effective Theory, double logarithms arise leading to $m \leq 2n$.

2. The Wilsonian effective action

Consider a field theory with characteristic large energy scale M, and suppose we are only interested in physics at low energies $E \ll M$. This is the physical situation effective field theories are designed to analyze. The full theory is defined in terms of a path integral. Everything we wish to know can be obtained from calculating the expectation values

$$\langle 0 | T \{ \phi(x_1) \dots \phi(x_n) \} | 0 \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, e^{iS(\phi)} \phi(x_1) \dots \phi(x_n), \tag{2.1}$$

where the integration measure is

$$\int \mathcal{D}\phi = \prod_{x^\mu} \int d\phi(x) \quad \text{or} \quad \int \mathcal{D}\phi = \prod_{p^\mu} \int d\tilde{\phi}(p), \tag{2.2}$$

and

$$Z = \int \mathcal{D}\phi e^{iS(\phi)}. \tag{2.3}$$

To obtain the low-energy effective action, we split the field

$$\phi = \phi_L + \phi_H, \qquad \int \mathcal{D}\phi = \int \mathcal{D}\phi_L \int \mathcal{D}\phi_H, \tag{2.4}$$

where ϕ_H contains all Fourier modes with $\omega \geq \Lambda$ and ϕ_L the low-energy modes $\omega < \Lambda$. Since we are only interested in low-energy physics, we only need to consider correlation functions

$$\langle 0 | T \{ \phi_L(x_1) \dots \phi_L(x_n) \} | 0 \rangle = \frac{1}{Z} \int \mathcal{D}\phi_L \underbrace{\int \mathcal{D}\phi_H \, e^{iS(\phi_L + \phi_H)}}_{e^{iS_\Lambda(\phi_L)}} \phi_L(x_1) \dots \phi_L(x_n) \tag{2.5}$$
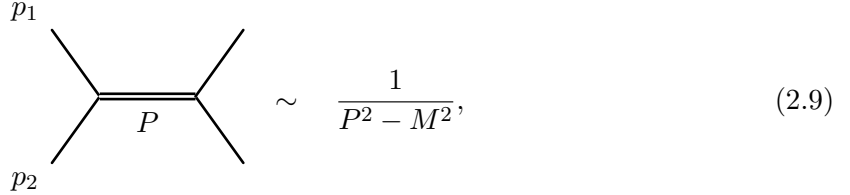
$$= \frac{1}{Z} \int d\phi_L \, e^{iS_\Lambda(\phi_L)} \phi_L(x_1) \dots \phi_L(x_n). \tag{2.6}$$

$S_\Lambda(\phi_L)$ is called the "Wilsonian effective action" and we have chosen $\Lambda \leq M$ to integrate out the physics associated with M. $S_\Lambda(\phi_L)$ is non-local on scales $\Delta x^\mu \gtrsim \frac{1}{\Lambda}$ (i.e., the Lagrangian is not just a polynomial of the fields or their derivatives evaluated at a single point in space-time), because high-energy fluctuations have been integrated out. As a final step, one expands the non-local action as a series of local operators. This expansion is possible because $E \ll \Lambda$ and is analogous to the multipole expansion in (1.2). We will encounter concrete examples below. The result has the form

$$S_\Lambda(\phi_L) = \int d^d x \, \mathcal{L}_\Lambda^{\text{eff}}(x), \tag{2.7}$$

$$\mathcal{L}_\Lambda^{\text{eff}}(x) = \sum_i g_i O_i(x), \tag{2.8}$$

where the object $\mathcal{L}_\Lambda^{\text{eff}}$ is called the effective Lagrangian. It is an infinite sum over local operators O_i allowed by symmetries. The coefficients g_i are referred to as Wilson coefficients. To make this a little more concrete, assume that we integrated out a heavy particle with mass M. The full theory might contain diagrams such as

$$\sim \quad \frac{1}{P^2 - M^2}, \tag{2.9}$$

where the two incoming lines represent the light field ϕ_L and the double line the heavy field ϕ_H. Since $p_1, p_2 \ll M$, we can expand

$$\frac{1}{p^2 - M^2} = -\frac{1}{M^2} - \frac{p^2}{M^4} + \cdots \rightarrow -\frac{1}{M^2}\delta^{(4)}(x) + \frac{\square}{M^4}\delta^{(4)}(x), \tag{2.10}$$

so $\mathcal{L}_\Lambda^{\text{eff}}$ will contain terms such as $\phi_L^4(x)$ and $\partial_\mu \phi_L(x)\partial^\mu \phi_L(x)\phi_L^2(x)$ etc. In general, it will be very hard to calculate the coefficients g_i and since we ended up with infinitely many terms in $\mathcal{L}_\Lambda^{\text{eff}}$ it is, a priori, unclear how the construction is useful.

The required ordering principle is provided by dimensional analysis. With $\hbar = c = 1$ $[m] = [E] = [x^{-1}] = [t^{-1}]$ all quantities measured in the same units. Assuming that $[g_i] = -\gamma_i$ is the mass dimension of g_i, it follows that

$$g_i = C_i M^{-\gamma_i}, \tag{2.11}$$

with a dimensionless coefficient C_i. Since the coefficients arose when integrating out the physics associated with M, it is natural to assume that $C_i = \mathcal{O}(1)$. Very large, e.g., $C_i \sim 10^6$, or very small coefficients, e.g., $C_i \sim 10^{-6}$, would call for some explanation in terms of degrees of freedom not considered. At low energy, the contribution of O_i to a dimensionless observable scales as

$$\left(\frac{E}{M}\right)^{\gamma_i} = \begin{cases} \mathcal{O}(1) & \gamma_i = 0 \\ \gg 1 & \gamma_i < 0 \\ \ll 1 & \gamma_i > 0 \end{cases}, \tag{2.12}$$

and therefore only operators with $\gamma_i \leq 0$ are important at low energy.

To derive the mass dimension δ_i of a given operator, we need to know the mass dimension of the fields. Assuming that the theory is weakly coupled, the scaling dimension is determined by the free action

$$S_0 = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2\right) \tag{2.13}$$

with $[x] = -1$, $[\partial_\mu] = +1 \sim E$, and using that the action is dimensionless, we find

$$[\phi] = \frac{d}{2} - 1, \qquad \text{i.e.} \quad \phi \sim E^{\frac{d}{2}-1}. \tag{2.14}$$

For an operator with mass dimension δ_i, we have $\gamma_i = \delta_i - d$, e.g.

	δ_i	γ_i	scaling of g_i
$\partial_\mu\phi\partial^\mu\phi$	d	0	1
ϕ^2	$d-2$	-2	M^2
ϕ^4	$2d-4$	$d-4$	M^{4-d}
$(\partial_\mu\phi)^2\phi^2$	$2d-2$	$d-2$	M^{2-d}
ϕ^6	$3d-6$	$2d-6$	M^{6-2d}

For an operator with n scalar fields and m derivatives we have

$$\delta_i = n\left(\frac{d}{2}-1\right)+m, \qquad \gamma_i = (n-2)\left(\frac{d}{2}-1\right)+m-2, \tag{2.15}$$

so only very few operators have $\gamma_i \leq 0$ (unless $d \leq 2$). The following terminology is commonly used:

Dimension		Importance for $E \to 0$	Terminology for operator
$\delta_i < d$	$\gamma_i < 0$	grows	relevant (super-renormalizable)
$\delta_i = d$	$\gamma_i = 0$	constant	marginal (renormalizable)
$\delta_i > d$	$\gamma_i > 0$	declines	irrelevant (non-renormalizable)

This terminology is not optimal. For example, it is interesting to search for the effects mediated by irrelevant operators, since they provide information on the physics at very high energies.

Moreover, our discussion makes it clear that renormalizability is sometimes overrated: usually one avoids irrelevant operators because they render a theory nonrenormalizable. However, once we admit that a theory is not valid up to infinitely large energies, then it is clear that it will contain also irrelevant operators. This is not a problem, because their contributions are suppressed by some large scale M, at which new physics enters. Renormalizable Lagrangians are so successful in describing our low-energy measurements because the relevant and marginal operators are the most important ones at low energies.

2.1. Relevant and irrelevant operators

Before we proceed further with the renormalization group, we give two examples of irrelevant operators

- The weak interaction is so weak at low energies because it is mediated by irrelevant operators such as $O = \bar{u}\gamma_\mu(1-\gamma_5)d\,\bar{l}\gamma^\mu(1-\gamma_5)\nu$. The fermion field scales as $\psi \sim E^{\frac{3}{2}}$, so $\delta = 6$, $\gamma = 2$. The coefficient of the operator must be proportional to $\frac{1}{M^2}$. Here, the mass $M = M_W$ is the mass of the W-boson. From the form of the interaction, Oskar Klein predicted the existence of massive particles with $M_W \geq 60$ GeV already in 1938.

- If there are only left-handed neutrinos – as is assumed in the Standard Model – we cannot write down a dimension-3 fermion mass term (see (2.19) below). However, the gauge symmetries of the Standard Model allow us to write down a dimension-5 term $g\,O = g\,\nu^T H H \nu$ with $g \sim \frac{1}{\Lambda}$ (ν and H are neutrino and Higgs fields, respectively). After electroweak symmetry breaking this yields a Majorana mass term for the neutrino, with $m_\nu \sim \frac{\langle H \rangle^2}{\Lambda}$ (where $\langle H \rangle \sim 174$ GeV is the vacuum expectation value of the Higgs field).

The fact that $\Lambda \sim 10^{14}$ GeV can be interpreted as evidence for physics beyond the Standard Model at these scales.

While irrelevant operators are perfectly natural, super-renormalizable/relevant operators are problematic. Consider for example the ϕ^2 operator in ϕ^4 theory. In $d = 4$, we have $\delta_i = 2$, $\gamma_i = -2$, and so we expect that $m^2 \sim \Lambda^2$. Integrating out the quantum fluctuations at large scales generates a large mass for scalar particles. But this is a contradiction: if $m^2 \sim \Lambda^2$ we should have integrated out the corresponding field ϕ. Note that also $\bar{\psi}\psi \sim E^3$ is relevant. This reasoning leads one to conclude that only theories whose mass terms are forbidden by symmetries are natural. Looking at the Standard Model as an effective theory, this condition is almost fulfilled:

- gauge bosons do not have mass terms because they are forbidden by gauge symmetry, i.e.

$$m^2(A_\mu)^2 \rightarrow m^2 e^{2i\alpha}(A_\mu)^2 \qquad (2.16)$$

  is not invariant,

- fermion fields do not have mass terms because left- and right-handed fields

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\psi, \qquad (2.17)$$

$$\psi_R = \frac{1}{2}(1 + \gamma_5)\psi, \qquad (2.18)$$

  have different gauge charges: ψ_R is neutral under $SU(2)_L$, ψ_L is not, so that a mass term

$$m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R) \qquad (2.19)$$

  would violate gauge invariance.

Note that the absence of a mass term in $\mathcal{L}_{\mathrm{SM}}$ does not imply that the fermions and gauge bosons are massless. They receive their mass by interacting with the Higgs condensate $\bar{\psi}_L H \psi_R \rightarrow \bar{\psi}_L \langle H \rangle \psi_R$, where $\langle H \rangle$ is the vacuum expectation value. It is remarkable, that there is only one relevant operator in the entire Standard Model Lagrangian: the mass term of the Higgs field $\mu^2 H^\dagger H$.

There are several ways out of this dilemma, but all of them involve physics beyond the Standard Model around the scale of the Higgs mass:

- Supersymmetry relates fermions and bosons. It can be used to protect scalar masses. Constructing the theory in such a way that fermion masses are forbidden then implies that scalar masses are also forbidden as well.

- In technicolor models, there is no fundamental Higgs boson. The Higgs particle is a bound state of a fermion–antifermion pair, similar to mesons in QCD.

- In little Higgs models, the Higgs boson is a pseudo-Goldstone boson of a spontaneously broken global symmetry.

Alternatively, the smallness of M_H could just be due to some accidental cancellation. To make this more plausible, people often invoke the anthropic principle: "If the Universe (in our example M_H) were much different, we would not be here." There is no concrete evidence for any of these explanations, and we will not revisit this more philosophical discussion in the following.

2.2. Renormalization Group

So far, we have considered a situation where we integrated out physics above some characteristic scale M. It is also interesting to look at what happens if we only integrate out a small slice $\Lambda > \omega > \Lambda - \delta\Lambda$ in which the particle content remains unchanged. In this case, the form of the action is unchanged, only the coefficients g_i change. Repeating the procedure, one obtains the couplings as a function of the cutoff

$$\{g_i(\Lambda)\} \to \{g_i(\Lambda - \delta\Lambda)\} \to \{g_i(\Lambda - 2\delta\Lambda)\} \to \dots \tag{2.20}$$

The evolution equations

$$\Lambda\frac{dg_i}{d\Lambda} = f(\{g_i\}) \tag{2.21}$$

are called renormalization group (RG) equations.[1]

We will first derive this RG evolution for the trivial but instructive case of a quadratic action and then analyze the case of ϕ^4 theory. The latter case is interesting because we can analyze the effect of quantum corrections on marginal operators.

2.2.1. Quadratic action

The general form of a quadratic action is

$$\mathcal{L} = \frac{1}{2}\phi(x)[-m^2 - 1 \times \Box + c\,\Box^2 + \dots]\phi(x)\,, \tag{2.22}$$

where we have used integration by part to move all derivatives on the first field to the second one to reduce the number of terms. This is possible since the physical fields should go to zero for very large x so that boundary terms do not contribute.

After Fourier transform

$$\phi(x) = \int_k e^{-ikx}\tilde{\phi}(k) = \int \frac{d^dk}{(2\pi)^d}e^{-ikx}\tilde{\phi}(k), \tag{2.23}$$

the action becomes

$$S = \frac{1}{2}\int d^dx \int_p \int_k \tilde{\phi}(p)[-m^2 + k^2 + ck^4 + \dots]\tilde{\phi}(k)e^{-i(p+k)x} \tag{2.24}$$

$$= \frac{1}{2}\int_k \tilde{\phi}(-k)[-m^2 + k^2 + ck^4 + \dots]\tilde{\phi}(k). \tag{2.25}$$

We further assume that our theory is defined with a UV cutoff Λ on each component of the momentum

$$\int_k \to \int_k^\Lambda = \int_{-\Lambda}^\Lambda \frac{dk^0}{2\pi} \int_{-\Lambda}^\Lambda \frac{dk^1}{2\pi} \cdots \int_{-\Lambda}^\Lambda \frac{dk^{d-1}}{2\pi}. \tag{2.26}$$

Let us now split $\phi = \phi_L + \phi_H$ according to

$$\tilde{\phi}(k) = \tilde{\phi}_L(k) + \tilde{\phi}_H(k) \tag{2.27}$$

$$= \begin{cases} \tilde{\phi}_L(k) & |k_\mu| < b\Lambda, \quad \forall\mu \\ \tilde{\phi}_H(k) & |k_\mu| > b\Lambda, \quad \text{for some } \mu \end{cases} \tag{2.28}$$

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[1] Integrating out momentum modes maps the general Lagrangian onto itself and one can perform several steps after one another, which inspired the name "group". However, the mapping does not need to be invertible, so the renormalization group is only a semi-group mathematically.

The field ϕ_L describes functions below $\Lambda' = b\Lambda$, with a free parameter $b \in [0,1]$. Our action splits accordingly into

$$S = S_L + S_H = \frac{1}{2}\int_k^{b\Lambda} \tilde{\phi}_L(-k)[\dots]\tilde{\phi}_L(k) + \frac{1}{2}\int_k^{\Lambda} \tilde{\phi}_H(-k)[\dots]\tilde{\phi}_H(k). \qquad (2.29)$$

If we are only interested in low energy Green's functions

$$\langle 0| T\{\phi_L(x_1)\dots\phi_L(x_n)\} |0\rangle = \frac{1}{Z}\int \mathcal{D}\phi_L \int \mathcal{D}\phi_H\, e^{iS_H} e^{iS_L} \phi_L(x_1)\dots\phi_L(x_n), \qquad (2.30)$$

then the effect of ϕ_H is entirely absorbed into the normalization, so that

$$\langle 0| T\{\phi_L(x_1)\dots\phi_L(x_n)\} |0\rangle = \frac{1}{Z_L}\int \mathcal{D}\phi_L\, e^{iS_L} \phi_L(x_1)\dots\phi_L(x_n). \qquad (2.31)$$

To directly compare S_L with S, let us now rescale

$$k' = \frac{k}{b}, \qquad x' = xb. \qquad (2.32)$$

In terms of the variable k', the cutoff moves back to Λ. The action becomes

$$S_L = \frac{1}{2}\int_{k'}^{\Lambda} b^d \tilde{\phi}(-bk')[-m^2 + b^2 k'^2 + b^4 c k'^4 + \dots]\tilde{\phi}(bk'). \qquad (2.33)$$

Let us further rescale $\tilde{\phi}(bk') \to \tilde{\phi}'(k') \times b^{-\frac{d+2}{2}}$ to have a canonically normalized kinetic term. The resulting action

$$S_L = \frac{1}{2}\int_{k'}^{\Lambda} \tilde{\phi}'(-k')[-\frac{m^2}{b^2} + k'^2 + b^2 c k'^4 + \dots]\tilde{\phi}'(k') \qquad (2.34)$$

shows that we get the same theory, but with

$$m^2 \to \frac{m^2}{b^2} \quad \text{(relevant)}, \qquad c \to b^2 c \quad \text{(irrelevant)}, \qquad (2.35)$$

so for $b = \frac{1}{2}$, for example, the mass becomes four times as large, while the coefficient of the four-derivative term is four times smaller. If we iterate the transformation (making b smaller and smaller), we get the *renormalization group flow* in the space of coupling constants:

When this analysis is extended to theories that include small couplings, the result is basically unchanged. The irrelevant operators remain irrelevant, and the relevant ones stay relevant. However, it becomes very interesting to check what happens with *marginal* operators. The small perturbation induced by the coupling will make them marginally relevant, or marginally irrelevant. For QCD, it turns out that the coupling slowly gets stronger as the high-energy modes are integrated out. Starting with an essentially free theory defined with a very high cutoff Λ, one ends up with a strongly coupled theory at low energy. This property is called asymptotic freedom. As we will show now, the situation is opposite for ϕ^4 theory. Even if the theory has a large coupling in its Lagrangian, it looks more and more like a free theory when the high-energy modes are integrated out.
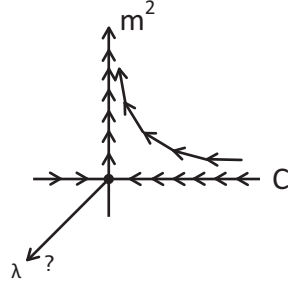
Figure 2.1.: The point $m = c = \cdots = 0$, the massless scalar field action, is a *fixed point*. This is called the Gaussian fixed point.

2.2.2. ϕ^4 theory

We now turn to ϕ^4 theory. In Minkowski space with coordinates x_M the associated Lagrangian takes the form

$$S_M = \int d^d x_M \left[\frac{1}{2}(\partial^M_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right],$$ (2.36)

but for our discussion, it is advantageous to do a Wick rotation i.e., write $t_M = -it_E$, $x_M^2 = -x_E^2$. Rewriting the Lagrangian, we obtain

$$S_M = i \int d^d x_E \left[\frac{1}{2}(\partial^E_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right] = iS_E.$$ (2.37)

In Euclidean space, time and space are treated in a symmetrical way, which allows us to impose the momentum cutoff on the absolute value $|k_E|^2 = k_E^2 = \vec{k}_E^2$ instead of the individual components as in (2.27). Remember that in Minkowski space the condition $k_M^2 = 0$ does not imply that the individual momentum components are small: the momentum could belong to an energetic massless particle.

The second important advantage of Euclidean space is that the path integral is much better behaved. Instead of an exponential with a fast-changing phase, large configurations of the action lead to exponential suppression. Dropping the subscript and writing $x^\mu \equiv x_E^\mu$, $k \equiv k_E^\mu$ the Euclidean path integral with a cutoff Λ takes the form

$$Z = \int \mathcal{D}\phi \, \exp\left[-\int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right) \right],$$ (2.38)

where $\mathcal{D}\phi = \prod_{|k|<\Lambda} d\tilde{\phi}(k)$. We now again split the field into a high- and a low-energy component

$$\tilde{\phi}(k) = \tilde{\phi}_L(k) + \tilde{\phi}_H(k),$$ (2.39)

where the two involve cutoffs

$$\tilde{\phi}_L(k) = \tilde{\phi}(k)\,\theta(|k| < b\Lambda),$$ (2.40)

$$\tilde{\phi}_H(k) = \tilde{\phi}(k)\,\theta(|k| > b\Lambda).$$ (2.41)

in addition to the overall cutoff $\theta(|k| < \Lambda)$ present in the original field.[2] The quadratic part of the action will again just turn into a sum of quadratic actions, but the interaction now also includes crossed terms:

$$S(\phi_L + \phi_H) = S(\phi_L) + S(\phi_H) + \int d^4x \, \lambda \left[\frac{\phi_L \phi_H^3}{3!} + \frac{\phi_L^2 \phi_H^2}{2!2!} + \frac{\phi_L^3 \phi_H}{3!} \right]. \tag{2.42}$$

Now we will derive the Feynman rules and then integrate over ϕ_H to lowest order in perturbation theory. The propagators in the free theory are

$$\Delta_L = \langle 0| \, T\{\phi_L(x)\phi_L(0)\} \, |0\rangle = \int^\Lambda \frac{d^4k}{(2\pi)^4} e^{ikx} \frac{1}{k^2 + m^2} \theta(|k| < b\Lambda), \tag{2.43}$$

$$\Delta_H = \langle 0| \, T\{\phi_H(x)\phi_H(x)\} \, |0\rangle = \int^\Lambda \frac{d^4k}{(2\pi)^4} e^{ikx} \frac{1}{k^2 + m^2} \theta(|k| > b\Lambda), \tag{2.44}$$

which we will denote as

$$\Delta_L = \text{———} \, , \tag{2.45}$$

$$\Delta_H = \text{═══} \, . \tag{2.46}$$

The mixed propagator

$$\langle 0| \, T\{\phi_H(x)\phi_L(x)\} \, |0\rangle = 0 \tag{2.47}$$

vanishes since there is no such term in the action because the conditions $\theta(|k| > b\Lambda)$ and $\theta(|k| < b\Lambda)$ are mutually exclusive. The Feynman rules for the interactions are
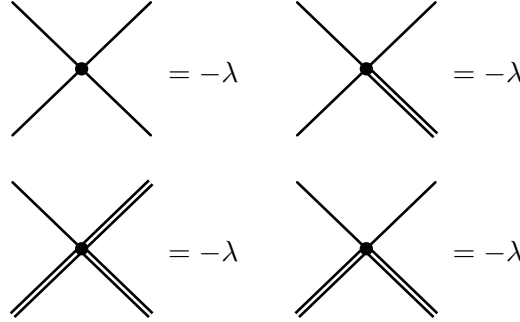
Note that the factorials in (2.42) neatly arrange themselves so that the n-th power of a given field is divided by $n!$, which cancels the factor from identical contractions.

At tree level, we have diagrams such as

$$= \frac{\lambda^2}{(p_1 + p_2 + p_3)^2 + m^2} \theta(|p_1 + p_2 + p_3| > \Lambda b), \tag{2.48}$$

which corresponds to a $\frac{1}{(\Lambda b)^2}$-suppressed ϕ_L^6 interaction after integrating out ϕ_H.

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[2] We use the notation $\theta(x < a) \equiv \theta(a - x)$

2. The Wilsonian effective action

However, we are mainly interested in the behavior of the ϕ^4 interaction. In this case, there is no tree-level contribution, but one-loop diagrams of the form
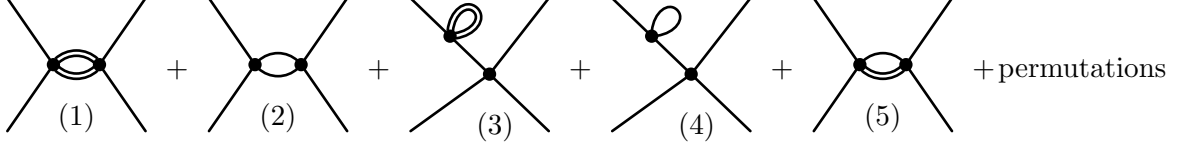

$(2) + (4)$ are diagrams in the low-energy theory, so we only need to evaluate (1), (3) and (5). It turns out that (3) only contributes to the mass term ϕ_L^2, not the interaction. Diagram (5) vanishes when the external momenta are set to zero since then the loop momentum no longer has any support. This diagram can therefore only contribute to higher-dimensional, irrelevant operators. This leaves diagram (1), for which we get

$$D_1 = \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \frac{1}{(k + p_1 + p_2)^2 + m^2} \theta(|k| > \Lambda b)\theta(|k| < \Lambda)$$
$$\times \theta(|k + p_1 + p_2| > \Lambda b)\theta(|k + p_1 + p_2| < \Lambda). \tag{2.49}$$

Since $m \ll \Lambda b$, $p_i^\mu \ll \Lambda b$ (the *external* momenta can be taken arbitrarily small), we can Taylor expand on the level of the integrand. Higher orders in the expansion are suppressed by $\frac{m}{\Lambda}$, $\frac{p^\mu}{\Lambda}$ and match onto irrelevant operators. Setting to momenta to zero to extract the contribution to the ϕ_L^4 interaction gives

$$D_1 = \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2} \theta(|k| > \Lambda b)\theta(|k| < \Lambda) + \ldots \tag{2.50}$$

$$= \frac{\lambda^2}{2} \frac{\Omega_d}{(2\pi)^d} \int_{b\Lambda}^{\Lambda} dk\, k^{d-5} \tag{2.51}$$

$$= \frac{\lambda^2}{2} \frac{\Omega_d}{(2\pi)^d} \frac{\Lambda^{d-4} - (\Lambda b)^{d-4}}{d - 4}, \tag{2.52}$$

with $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ and

$$\lim_{d \to 4} \frac{\Lambda^{d-4} - (\Lambda b)^{d-4}}{d - 4} = \lim_{d \to 4} \frac{1 - b^{d-4}}{d - 4} = -\log b. \tag{2.53}$$

Further, for $d = 4$ we have $\Omega_4 = 2\pi^2$ and thus

$$D_1 = \frac{\lambda^2}{16\pi^2} \log \frac{1}{b}. \tag{2.54}$$

Accounting also for the crossed diagrams and yields the result

 $= \frac{3\lambda^2}{16\pi^2} \log \frac{1}{b} + \mathcal{O}(\lambda^4). \tag{2.55}$
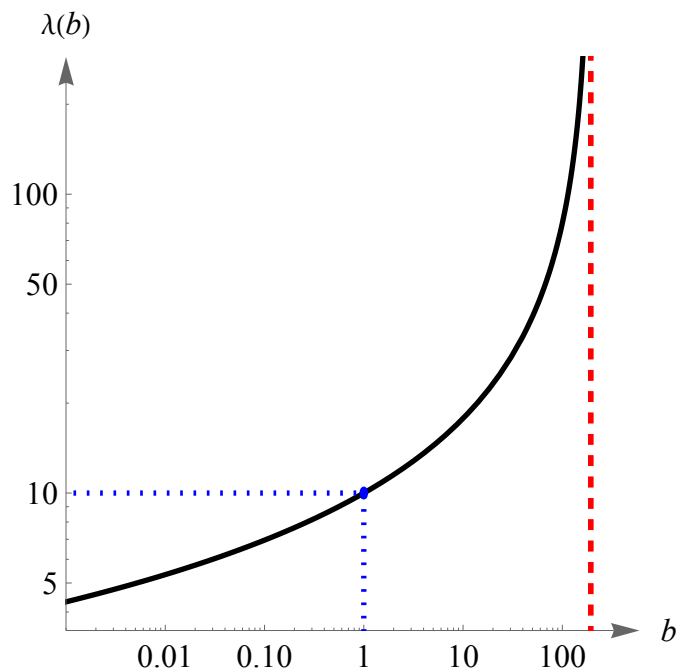
Figure 2.2.: The coupling as a function of the parameter b (black line), for the choice $\lambda = 10$ for $b = 1$ (blue dotted lines). The red, dashed line shows the Landau pole, i.e. the value of b where at which the coupling becomes infinite in the one-loop approximation (2.59). Note that the coupling goes to zero as $b \to 0$.

In the low-energy theory, this contribution must arise from $-\int d^4x \frac{\lambda'}{4!}\phi_L^4$, so we can identify

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\log\frac{1}{b} \tag{2.56}$$

The coupling gets *weaker* when high-energy modes are integrated out! Let us imagine that we integrate the high-energy physics little by little, so integrating

$$d\lambda = +\frac{3\lambda^2}{16\pi^2}d\log b \tag{2.57}$$

we find

$$\int_{\lambda(1)}^{\lambda(b)} \frac{d\lambda}{\lambda^2} = \frac{-1}{\lambda(b)} + \frac{1}{\lambda(1)} = \frac{3}{16\pi^2}\log b, \tag{2.58}$$

and thus

$$\lambda(b) = \frac{\lambda(1)}{1 + \frac{3}{16\pi^2}\lambda(1)\log\frac{1}{b}}. \tag{2.59}$$

Our analysis was a bit simplistic, in that we only looked at one operator, ϕ^4, and did not include the effects of irrelevant operators. Also, our conclusion is only valid at small coupling since we computed our result to the lowest order in an expansion in the coupling constant. However, all available (perturbative and non-perturbative) evidence suggests that

the behavior persists at arbitrary values of λ. Since the coupling becomes small at low energies, one needs to start with a sufficiently strong coupling at large values of the cutoff

$$\lambda(1) = \frac{\lambda(b)}{1 - \frac{3}{16\pi^2}\lambda(b)\log\frac{1}{b}}. \tag{2.60}$$

Since $\lambda(1) \to \infty$ for $\log\frac{1}{b} = \frac{16\pi^2}{3}\frac{1}{\lambda(b)}$ our perturbative analysis suggests that the cutoff cannot be arbitrarily large, $\log\frac{1}{b} \sim \log\frac{\Lambda}{E} \leq \frac{16\pi^2}{3}\frac{1}{\lambda(b)}$.

Since we did our analysis in perturbation theory, the extrapolation $\lambda \to \infty$ is not meaningful, but a non-perturbative analysis leads to the same result: if one takes the cutoff too large, one ends up with a free theory at low energies. Theories that have the property that the cutoff cannot be chosen arbitrarily large for non-vanishing λ are called trivial. All evidence strongly suggests that $\lambda\phi^4$ is a trivial theory. As in the case of irrelevant operators, the term trivial is not optimal. What it implies is that a theory like ϕ^4 cannot be valid to infinitely large energies and must be interpreted as a low-energy effective theory.

Historically, the discovery by Landau and collaborators that the coupling (2.60) grows at large energies and seems to explode at large, but finite energy made people skeptical about quantum field theory as a framework for fundamental physics. The scale at which the coupling diverges in the approximation (2.60) depends exponentially on the value of the coupling. Denoting this scale by Λ and the low-energy scale by $\Lambda_0 = b\Lambda$, the pole occurs at

$$\Lambda = \Lambda_0 \exp\left(\frac{16\pi^2}{\lambda(b)}\right), \tag{2.61}$$

so for a weakly coupled theory at low energies, the *Landau pole* arises at a very high energy. In QED, $\Lambda \sim 10^{277}$ GeV, much higher than the Planck scale $\Lambda \sim 10^{19}$ GeV, so in practice the presence of a cutoff is not a problem. For some time, people believed that *all* QFTs would have this behavior. It was then found that there is one exception, namely non-abelian gauge theories which have the opposite behavior. In these theories, the coupling becomes weaker at high energies, a property called *asymptotic freedom*.

To finish this chapter, let us introduce a commonly used term. In our operator discussion, we considered Wilson coefficients g_i of dimension γ_i which scale as

$$g_i = C_i \Lambda^{-\gamma_i} \tag{2.62}$$

If the coefficient C_i would be Λ independent, the logarithmic derivative would simply give the dimension

$$\Lambda\frac{d}{d\Lambda}\ln g_i = -\gamma_i \tag{2.63}$$

of the coupling. The ϕ^4 coupling λ is dimensionless, i.e. it has scaling dimension $\gamma = 0$, but above we found that it nevertheless has a logarithmic dependence on the cutoff Λ

$$\Lambda\frac{d}{d\Lambda}\ln\lambda = \frac{3}{16\pi^2}\lambda + \mathcal{O}(\lambda^3) \tag{2.64}$$

This additional logarithmic Λ dependence of the dimensionless coefficients C_i is generated by quantum effects and is called the *anomalous dimension* of the operator.

3. Continuum effective theory

The construction of the Wilsonian effective action is physically very intuitive, and leads to a new perspective on renormalization. However, actually integrating out the physics above a cutoff Λ is typically as difficult as solving the theory.

Instead of explicitly integrating out the high-energy physics, it is much easier to simply construct the most general low-energy theory, i.e. to use a bottom-up approach in constructing the effective Lagrangian rather than a top down approach based on computing a complicated path integral. Also, it is much simpler to work without a hard cutoff and to treat the effective theory like a standard continuum field theory. To get the effective theory in this bottom-up approach, one follows a number of steps, which we now discuss in turn:

1. Identify the low-energy degrees of freedom. This can be simple: e.g., if we consider a theory with a very heavy particle and weak coupling, the low-energy degrees of freedom are simply all light particles. In other cases it is not trivial: in QCD, the low-energy degrees of freedom are pions, kaons, protons, neutrons, etc. and not the quarks and gluons in the high-energy Lagrangian.

2. Introduce a field for each degree of freedom and construct the most general low-energy $\mathcal{L}_{\text{eff}}$ consistent with the symmetries of the full theory. Order the operators in $\mathcal{L}_{\text{eff}}$ by their dimension.

3. Matching. To determine the coupling constants in $\mathcal{L}_{\text{eff}}$ calculate a number of correlation functions (or scattering amplitudes) in both the full and the effective theory. Expand the full theory result around the low-energy limit and adjust the Wilson coefficients in $\mathcal{L}_{\text{eff}}$ in such a way that the full and EFT results agree.

4. RG improvement. The perturbative expansion of the Wilson coefficients can be improved by using RG equations for the coefficients.

It is simplest to use dimensional regularization (and the $\overline{\text{MS}}$ scheme) in both the full and the effective theory. At first sight it seems troubling to work without a hard cutoff and integrate out to arbitrarily high energies even in the low-energy theory, which is not valid at high energies. However, we know from Wilson that we can absorb arbitrary high-energy physics into the couplings of $\mathcal{L}_{\text{eff}}$. By adjusting the couplings, we can thus obtain the correct low-energy results despite the incorrect behavior of our amplitudes at high energies.

Let us use a toy model with a heavy and light scalar field to illustrate the above steps. Our full theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_L)^2 - \frac{m^2}{2}\phi_L^2 + \frac{1}{2}(\partial_\mu \phi_H)^2 - \frac{M^2}{2}\phi_H^2$$
$$- \frac{\lambda_L}{4!}\phi_L^4 - \frac{\lambda_H}{4!}\phi_H^4 - \frac{\lambda_{HL}}{4}\phi_L^2\phi_H^2 - \frac{g}{2}\phi_H\phi_L^2. \tag{3.1}$$

Note that $\mathcal{L}$ is symmetric under $\phi_L \to -\phi_L$. To renormalize the theory (i.e. absorb all divergences of loop diagrams) we need to include also the terms

$$\delta\mathcal{L} = A + B\phi_H + C\phi_H^3, \tag{3.2}$$

but we assume that A, B, C are renormalized to zero. These terms will not be relevant for the discussion. Now let us follow the different steps to construct $\mathcal{L}_{\text{eff}}$:

1. The low-energy degrees of freedom for $E \ll M$ are just the light scalar particles described by the field ϕ_L. We will denote the effective theory field by ϕ_L like the corresponding full theory field. However, all that matters at this point is that the low energy theory contains a light scalar field with a $\phi_L \to -\phi_L$ symmetry, for which we well construct a Lagrangian.

2. The effective Lagrangian takes the form

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \phi_L)^2 - \frac{\tilde{m}^2}{2}\phi_L^2 - \frac{\tilde{\lambda}}{4!}\phi_L^4 - \frac{1}{2}\frac{C_{2,4}}{M^2}\phi_L\Box^2\phi_L \tag{3.3}$$

$$- \frac{1}{6!}\frac{C_{6,0}}{M^2}\phi_L^6 - \frac{1}{4!}\frac{C_{4,2}}{M^2}\phi_L^2\Box\phi_L^2 + \mathcal{O}\left(\frac{1}{M^4}\right), \tag{3.4}$$

where we only included operators up to dimension $\delta = 6$ and used the $\phi_L \to -\phi_L$ symmetry to exclude odd powers of the field. To show that all other $d = 6$ operators reduce to these three one uses integration by parts and drops total derivatives. Note that the coefficients of the operators in $\mathcal{L}_{\text{eff}}$ are different than the ones in the full Lagrangian (3.1). We already know from Wilson effective theory that the couplings (or Wilson coefficients) change when one integrates out high-energy physics, so $\tilde{m} \neq m$, etc.

3. Matching. To extract the values of $\tilde{m}$, $\tilde{\lambda}$, $C_{6,0}$, and $C_{4,2}$ we will now calculate the two-, four-, and six-point functions. We will show below that the effect of $C_{2,4}$ can be removed via a field redefinition if one only wants to reproduce physical quantities at low energies (rather than the full off-shell Green's functions), see Sec. 3.2.

3.1. Matching

In this section we will perform a set of tree-level computations in both the full and effective theory and adjust the couplings in $\mathcal{L}_{\text{eff}}$ so that the effective theory reproduces the full theory results. This should be possible, since our effective Lagrangian is the most general Lagrangian for the given degrees of freedom. Since we included all operators up to dimension $\delta = 6$ our Lagrangian will reproduce the full results up to corrections from $\delta = 8$ operators which are suppressed by $1/M^4$.

To perform the matching we should compute the same quantity in the full and the effective theory. The simplest possibility is to compute the time-ordered n-point Green's functions

$$G(x_1, x_2, \ldots, x_n) = \langle 0| T\{\phi(x_1) \ldots \phi(x_n)\} |0\rangle \tag{3.5}$$

introduced in (2.1). We will do the computations in momentum space and work with the Fourier transform $G(p_1, p_2, \ldots, p_n)$ of (3.5). It is furthermore convenient to work with the truncated Green's functions Γ:

$$(2\pi)^d \delta^{(d)}(p_1 + \cdots + p_n)\,\Gamma(p_1, p_2, \ldots, p_n) = G(p_1, p_2, \ldots, p_n) \prod_{i=1}^{n}\frac{1}{G(p_i)}, \tag{3.6}$$
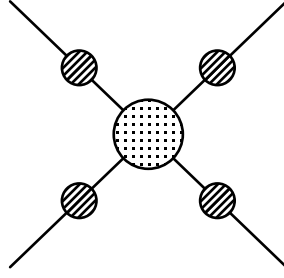
Figure 3.1.: Structure of the four-point function $G(p_1, p_2, p_3, p_4)$. The contributions the external lines give rise to full two-point functions $G(p_i)$ (shaded circles). The remaining part corresponds to the truncated 4-point function $\Gamma(p_1, p_2, p_3, p_4)$ (dotted circle in the center).

which are obtained by dividing out a full propagator for each leg and factoring out the overall momentum conservation delta function. The propagator is related to the two-point function by

$$G(p_1) = \int d^d x_1 \, e^{ip_1 x_1} \, \langle 0 | \, T\{\phi(x_1)\phi(0)\} \, | 0 \rangle \tag{3.7}$$

so that

$$G(p_1, p_2) = (2\pi)^d \delta^{(d)}(p_1 + p_2) \, G(p_1) \,. \tag{3.8}$$

If the effective theory reproduces the two-point function and the truncated Greens functions, it reproduces all Green's functions. For the four-point function, the relation between the full and the truncated Green's function is illustrated in Figure 3.1.

It is not really necessary that the effective theory should reproduces all Green's functions of the full theory. It is sufficient that physical quantities are the same in both theories. The Green's functions are not physical since they depend on the normalization of the field. A sufficient requirement is that the effective theory reproduces the scattering amplitudes $\mathcal{M}$, which are obtained from the truncated Green's functions through

$$i\mathcal{M}(p_1, \ldots, p_n) = \left(\sqrt{Z}\right)^n \Gamma(p_1, p_2, \ldots, p_n) \tag{3.9}$$

for on-shell momenta p_i after multiplying with the on-shell wave-function renormalization constants obtained from

$$\langle 0 | \, \phi(0) \, | p \rangle = \sqrt{Z} \,, \tag{3.10}$$

where $|p\rangle$ is a one-particle state. It is easy to verify that $\mathcal{M}(p_1, \ldots, p_n)$ is independent of the normalization of the field ϕ. We note that only the connected part of Γ contributes to the scattering amplitude, so we can replace Γ with its connected part on the right-hand side of (3.9). Also for the matching, we can restrict ourselves to reproducing all connected Green's functions, since the disconnected ones are just products of lower-point functions. The relation (3.9) between scattering amplitudes and truncated, connected Green's functions is called the Lehmann–Symanzik–Zimmermann (LSZ) reduction formula. For its derivation, we refer the reader to QFT books.

3. Continuum effective theory

Effective theory computation

The two-point function is just the inverse of the propagator

$$i\Gamma_2 = iG^{-1}(p) = p^2 - \tilde{m}^2 - \frac{C_{2,4}}{M^2}(p^2)^2 \,. \tag{3.11}$$

The two-point function is obtained from the Fourier transform of quadratic part of the action, up to the factor $1/2$ in the Lagrangian which gets cancelled because there are two equivalent contractions of the fields. The four-point function corresponds to the scattering of four identical particles, leading to

$$= \tilde{\lambda} - \frac{C_{4,2}}{M^2}\frac{1}{3}\left[(p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2\right], \tag{3.13}$$

where the result for the diagram with $C_{4,2}$ is derived in Appendix B. The last diagram corresponding to an external leg correction due to $C_{2,4}$ does not contribute to the truncated Green's function. The six-point function involves a contact term proportional to $C_{6,0}$ and diagrams composed of two insertions of $\tilde{\lambda}$, which we will not need to consider:

Full theory computation

The propagator in the full theory is

$$i\Gamma_2 = iG^{-1}(p) = (p^2 - m^2), \tag{3.15}$$

so that

$$\tilde{m} = m, \qquad C_{2,4} = 0, \tag{3.16}$$

where the second condition arises because $C_{2,4}$ leads to a p^4 term in the two-point function, which is not present at tree level order in the full theory. Please note that the relations (3.16) only hold a tree level and will be modified by loop corrections. The tree-level four-point function is given by

$$= \lambda_L + i(-ig)^2\left[\frac{i}{(p_1+p_2)^2 - M^2} + \frac{i}{(p_1-p_3)^2 - M^2} + \frac{i}{(p_1-p_4)^2 - M^2}\right]. \tag{3.18}$$

Since $p_i \ll M$ at low energies, we can Taylor expand Γ_4, leading to

$$i\Gamma_4 = \lambda_L - \frac{3g^2}{M^2} - \frac{g^2}{M^4}\left[(p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2\right]. \tag{3.19}$$

Comparison with the EFT result gives our first non-trivial matching condition

$$\tilde{\lambda} = \lambda_L - \frac{3g^2}{M^2}, \qquad C_{4,2} = \frac{3g^2}{M^2}. \tag{3.20}$$

Finally, we have the matching of the six-point function, involving the diagrams

$$i\Gamma_6 = \quad\qquad + \quad\qquad + \ldots \tag{3.21}$$

$$+ \quad\qquad + \quad\qquad + \ldots \tag{3.22}$$

The diagrams in the first line are one-particle-reducible with respect to the light field ϕ_L. These diagrams are automatically reproduced since we matched the four-point function. Only the diagrams in the second line will contribute to the matching on C_6. Since the operator does not involve derivatives it is sufficient to compute Γ_6 for vanishing momenta ($p_i = 0$). The resulting matching condition is

$$\frac{C_{6,0}}{M^2} = i(-ig)^2(-i\lambda_{HL})\left(\frac{i}{-M^2}\right)^2 \times 90 \tag{3.23}$$

so that the Wilson coefficient is given by

$$C_{6,0} = 90\frac{g^2\lambda_{HL}}{M^2}. \tag{3.24}$$

Note that each of the diagrams in the second line comes with a combinatorical factor of 1, since the vertices in (3.1) include $1/n!$ factors if they involve a given field to the n-th power. However there are many different diagrams, which are obtained from the first one by permuting external legs. The combinatorial factor in (3.24) is given as $90 = 6! \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2}$, because two permutations each at the two $\phi_L^2\phi_H$ and the $\phi_L^2\phi_H^2$ vertices are identical. This completes the construction of the effective theory at tree level.

3.2. Field redefinitions

With our matching computation we ensured that our effective theory reproduces the full theory result for the off-shell Green's functions. However, if we are only interested in physical quantities, such as scattering amplitudes, we can simplify the Lagrangian using field redefinitions. After all, the fields are just integration variables in the path integral and we should be able to change variables in the integral. As an example, consider

$$\phi_L \to \left[1 + \frac{\alpha}{M^2}\Box\right]\phi_L. \tag{3.25}$$

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Inserting this into $\mathcal{L}_{\text{eff}}$ and neglecting $\frac{1}{M^4}$ terms, we get

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}_{\text{eff}} - \frac{\alpha}{M^2}\phi_L\Big(\Box + m^2 + \frac{\tilde{\lambda}}{3!}\phi_L^2\Big)\Box\phi_L, \tag{3.26}$$

where we used an integration by parts for the kinetic term. By choosing $\alpha = -\frac{1}{2}C_{2,4}$, we can cancel the term $-\frac{1}{2}\frac{C_{2,4}}{M^2}\phi\Box^2\phi$ in $\mathcal{L}_{\text{eff}}$ so that $\mathcal{L}_{\text{eff}}'$ no longer contains this term:

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}_{\text{eff}}|_{C_{2,4}\to 0} + \frac{C_{2,4}}{2M^2}\phi_L(m^2 + \frac{\tilde{\lambda}}{3!}\phi_L^2)\Box\phi_L. \tag{3.27}$$

and the extra contributions correspond to operators which are already accounted for in the Lagrangian, namely $\phi_L\Box\phi_L$ and $\phi_L^3\Box\phi_L$.

An efficient short-cut to derive the effect of the field redefinition it to use the leading-power classical equation of motion (EOM)

$$\Big(\Box + m^2 + \frac{\tilde{\lambda}}{3!}\phi_L^2\Big)\phi_L = 0 \tag{3.28}$$

in order to eliminate higher-power terms in the Lagrangian. However, one should keep in mind that the field redefintion is performed in the quantum theory and does not rely on the classical limit.

In general, performing field redefinitions

$$\phi \to \phi + \Big(\frac{1}{M^2}\Big)^n f(\phi) = \phi + \delta\phi \tag{3.29}$$

generates

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}_{\text{eff}} + \underbrace{\Big(\frac{1}{M^2}\Big)^n f(\phi)}_{\delta\phi}\underbrace{\Big[\Box\phi + m^2\phi + \frac{\tilde{\lambda}}{3!}\phi^3\Big]}_{\text{EOM, from } \frac{\delta S}{\delta\phi}} + \mathcal{O}\Big(\Big(\frac{1}{M^2}\Big)^{n+1}\Big), \tag{3.30}$$

and allows one to systematically eliminate EOM terms from $\mathcal{L}_{\text{eff}}$. The reason that the classical EOM appears is that one can interpret the redefinition as a variation of the field, see (3.29) and that one can linearize in the variation since the higher order terms in $\delta\phi$ are power suppressed. Also, due to the suppression of $\delta\phi$ by $1/M^{2n}$, it is sufficient to consider the EOM of the leading power effective Lagrangian (3.28) since the contribution generated by irrelevant operators in the EOM are suppressed by additional powers of M^2.

In practice, one can iteratively perform as set of field redefintions to eliminate all EOM terms order-by-order in $1/M^2$. To this end, one first perform a general field redefintion (3.29) with $n = 1$. This can be used to eliminate all $1/M^2$ EOM operators in $\mathcal{L}_{\text{eff}}$. Next, one performs a general redefinition with $n = 2$. This will leave the $1/M^2$ operators unchanged but eliminates the EOM terms at $1/M^4$, and so forth. The upshpot is that one can always eliminate all EOM terms to any desired order.

The field redefinitions can reduce the number of operators in $\mathcal{L}_{\text{eff}}$ and they leave the physics content of the theory unchanged. This is true because:

1. ϕ and ϕ' have the same quantum numbers, so after inserting states

$$\lim_{x^0 \to \infty} \langle 0| T\{\phi(x_1) \ldots\} |0\rangle = \sum_X \langle 0| \phi(x_1) |X\rangle \langle X| T\{\ldots\} |0\rangle \qquad (3.31)$$

the same amplitudes can be extracted from the theory. The only thing that will change are the Z-factors, which for one-particle states $|p\rangle$ are given by $\langle 0| \phi(0) |p\rangle = Z^{1/2}$.

2. The Jacobian $\det\left(\frac{\delta\phi}{\delta\phi'}\right)$ is trivial, at least in dimensional regularization (see below).

Let us illustrate the first point using an explicit computation of a scattering amplitude before and after a field redefinition. For simplicity, we will do our computation using standard ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\tilde{m}^2}{2}\phi^2 - \frac{\tilde{\lambda}}{4!}\phi^4, \qquad (3.32)$$

The tree-level $2 \to 2$ scattering amplitude is

$$\mathcal{M} = -i\Gamma_4\left(\sqrt{Z}\right)^4 = \qquad = -\tilde{\lambda}, \qquad (3.33)$$

where we used that the $Z = 1$ at tree level. Let us now calculate this amplitude after the field redefinition $\phi \to \left(1 + \frac{\alpha}{M^2}\Box\right)\phi$.

$$\mathcal{L}' = \mathcal{L} - \frac{\alpha}{M^2}\phi\left(\Box + m^2 + \frac{\tilde{\lambda}}{3!}\phi^2\right)\Box\phi, \qquad (3.34)$$

which gives

$$\mathcal{M} = \qquad + \qquad + \ldots + \qquad$$

external leg correction,
removed by truncation

$$= -\tilde{\lambda}\left(1 - \frac{\alpha}{M^2}\frac{4!}{3!} \times \frac{1}{4}\underbrace{\sum_{i=1}^{4} p_i^2}_{4m^2}\right)\left(\sqrt{Z}\right)^4$$

$$= -\tilde{\lambda}\left(1 - \frac{\alpha}{M^2}4m^2\right)\left(\sqrt{Z}\right)^4. \qquad (3.35)$$

See Appendix B for details how to derive the Feynman rule for the $\phi^3\Box\phi$ term in (3.34). To get the Z-factor, it is easiest to evaluate the two-point function. Inserting one-particle states in the definition (3.7), one can show that they lead to a pole in the propagator

$$G(p) = \frac{Z}{p^2 - m_{\text{ph}}^2} + \text{"non-pole"}, \qquad (3.36)$$

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when p^2 is equal to the physical mass m_{ph}^2, whose residuum is the Z-factor. Evaluating the contribution of the extra terms in the Lagrangian, we get

$$\rule[0.5ex]{2cm}{0.5pt} \;+\; \rule[0.5ex]{0.8cm}{0.5pt}\blacksquare\rule[0.5ex]{0.8cm}{0.5pt} \;=\; \frac{i}{p^2-m^2} - i\frac{2\alpha}{M^2}\frac{i}{p^2-m^2}(-p^2)(-p^2+m^2)\frac{i}{p^2-m^2}$$

$$= \frac{i}{p^2-m^2}\left(1+\frac{2\alpha}{M^2}p^2\right) = \frac{i(1+\frac{2\alpha}{M^2}m^2)}{p^2-m^2} + \text{"non-pole"}, \quad (3.37)$$

We see that the extra contributions did not change the position of the pole in the propagator denominator, $m_{\text{ph}} = m$, but changed the residue so that the on-shell wave-function renormalization is given by

$$Z = 1 + \frac{2\alpha}{M^2}m^2. \tag{3.38}$$

For the scattering amplitude, we thus have

$$\mathcal{M} = -\tilde{\lambda}\left(1 - \frac{\alpha}{M^2}4m^2\right)\left(1 + \frac{2\alpha}{M^2}m^2\right)^2 = -\tilde{\lambda} + \mathcal{O}\left(\frac{1}{M^4}\right), \tag{3.39}$$

precisely as before the field redefinition (to the order considered). It is illustrative to see how the general statement 1.) is realized in the explicit computation.

Finally, let us derive statement 2.), namely that the Jacobian is trivial in dimensional regularization. As usual, the Jacobian is given by a determinant

$$\int \mathcal{D}\phi = \int \mathcal{D}\phi'\,\det\left(\frac{\delta\phi}{\delta\phi'}\right) \tag{3.40}$$

In our case, $\phi \to \phi' + \left(\frac{1}{M^2}\right)^n f(\phi')$ and since we perform functional integration, the jacobian is a functional derivative

$$\frac{\delta\phi}{\delta\phi'} = \delta(x-x') + \left(\frac{1}{M^2}\right)^n f'(\phi'(x))\,\delta(x-x'). \tag{3.41}$$

It is a local function since we only considered local variable changes, i.e. $\phi'(x)$ is only a function of $\phi(x)$, the function at the same point. A standard trick to get an explicit representation of a determinant is to write is as a Gaussian functional integral

$$\det\left(\frac{\delta\phi}{\delta\phi'}\right) = \int \mathcal{D}c \int \mathcal{D}\bar{c}\,\exp\left[i\int d^d x \int d^d y\,\bar{c}(x)\frac{\delta\phi}{\delta\phi'}c(y)\right]$$

$$= \int \mathcal{D}c \int \mathcal{D}\bar{c}\,\exp\left[i\int d^d x\,\bar{c}(x)\left[1+\left(\frac{1}{M^2}\right)^n f'(\phi')\right]c(x)\right], \tag{3.42}$$
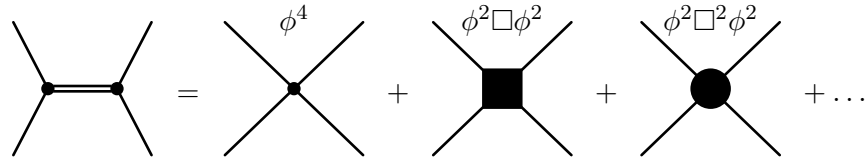
where c and $\bar{c}$ are Grassmann fields. Since these fields do not describe physical particles they are sometimes called ghost-fields.

Since the term involving f' is suppressed by $\frac{1}{M^2}$, it can be treated as a perturbation. The corresponding ghost diagrams are loops of a "fermion" with "propagator" $\frac{i}{1}$ (i.e., the ghost fields do not propagate since they do not have a kinetic term). Such loops that do not involve a scale, e.g., $\int d^d k \left(\frac{i}{1}\right)^n$, and vanish in dimensional regularization, hence $\det(\frac{\delta\phi}{\delta\phi'}) = 1$.

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3.3. Matching at higher orders

The general method to perform the matching is always the same: compute the same quantity both in the EFT and in the full theory and adjust the Wilson coefficients ("coupling constants") in the EFT in such a way that the results agree to the given order of the low-energy expansion.

At the tree level, it is simple to understand that the procedure works: the heavy particles are always far off-shell and their propagators get expanded into a polynomial. Since $\mathcal{L}_{\text{eff}}$ contains all higher-dimensional operators, it generates the most general polynomial contribution to the amplitude and by adjusting the couplings, we reproduce the full theory result. Schematically:

$$\frac{1}{p^2 - M^2} = -\frac{1}{M^2} - \frac{1}{M^2}\frac{p^2}{M^2} - \frac{1}{M^2}\frac{(p^2)^2}{M^4} - \dots \tag{3.43}$$

At higher orders, there are several interesting new features that emerge from the loop diagrams. Let us consider the self-energy diagram in the full theory
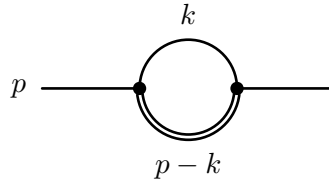
The following complications arise:

1. The loop momentum k can be small or large: the heavy propagator is *not* always off shell.

2. Renormalization: there are UV divergences both in the full and effective theory, and we thus have to renormalize all couplings/Wilson coefficients.

3. Loop diagrams are non-trivial (i.e., non-analytic) functions of external momenta and masses, but the Wilson coefficients C_i can only depend on the high-energy scale M ($\mathcal{L}_{\text{eff}}$ must be local). Accordingly, the non-trivial parts of the amplitudes must be present in the EFT loop computation, as they cannot be obtained from the C_i. In the end, this has to work out because the nontrivial dependence on the momenta is due to low-energy dynamics which is identical for $\mathcal{L}_{\text{full}}$ and $\mathcal{L}_{\text{eff}}$, but it will be illustrative to verify this in an explicit example.

4. Due to renormalization in $\mathcal{L}_{\text{eff}}$, the Wilson coefficients depend on the renormalization scale μ: $C_i = C_i(\mu)$. The renormalized coefficients involve terms of the form $g^2 \log\frac{\mu}{M}$, $\lambda_{HL} \log\frac{\mu}{M}$ etc.

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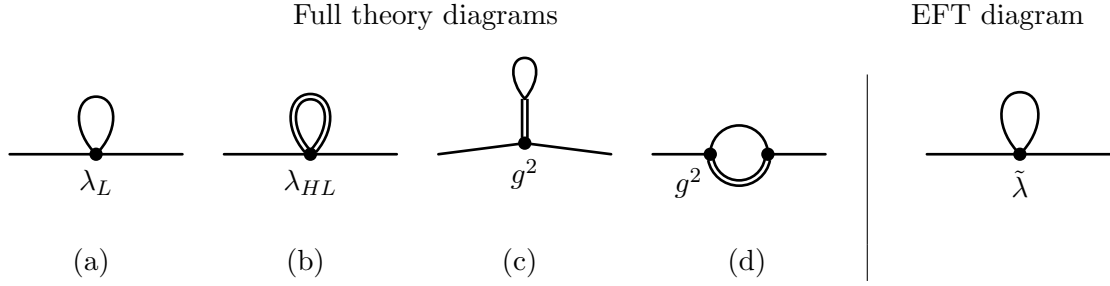
Figure 3.2.: One-loop corrections to the two-point function. In the full theory the two-point function gets contributions from the four diagrams (a) – (d), while there is only a single diagram contributing in the effective theory.

To see how all this works in practice, we will now perform a one-loop matching computation in the simplest possible setting: we will compute the two-point function and the one-loop corrections to $\tilde{m}$, the mass of the light particle in $\mathcal{L}_{\text{eff}}$. The relevant diagrams are shown in Figure 3.2.

The perturbative result for the two-point function takes the following form

$$
\begin{aligned}
&= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}(-i\Sigma)\frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}(-i\Sigma)\frac{i}{p^2 - m^2}(-i\Sigma)\frac{i}{p^2 - m^2} + \dots \\
&= \frac{i}{p^2 - m^2 - \Sigma}\,,
\end{aligned}
\tag{3.44}
$$

where the quantity Σ will in general depends on the momentum p^2 and corresponds to the one-particle irreducible contributions, i.e. diagrams which cannot be cut apart by cutting a single propagator. Below, we will compute Σ and match the results from the full and effective theory. The quantity Σ is closely related to the truncated two-point Green's function

$$
i\Gamma(p) = iG^{-1}(p)G(p)G^{-1}(p) = iG^{-1}(p) = p^2 - m^2 - \Sigma\,.
\tag{3.45}
$$

Full theory computation

For the above definition Σ is obtained by computing the blob in (3.44) corresponding to $(-i\Sigma)$ and multiplying by i to obtain Σ. The contribution of diagram (a) is thus given by

$$
\Sigma^{(a)} = i(-i\lambda_L\,\mu^{2\epsilon})\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m^2}\,,
\tag{3.46}
$$

where the factor one half is the symmetry factor and $\mu^{2\epsilon}$ is the *renormalization scale*, introduced to make λ dimensionless in $d = 4 - 2\epsilon$. Using the results from Appendix A we find

$$
\begin{aligned}
\Sigma^{(a)} &= \frac{\lambda_L}{2}(4\pi)^{-\frac{d}{2}}\Gamma\left(1 - \frac{d}{2}\right)(m^2)^{\frac{d}{2}-1}\mu^{2\epsilon} \\
&= \frac{m^2\lambda_L}{32\pi^2}\left[-\frac{1}{\epsilon} + \gamma_E - \log(4\pi) - 1 + \log\frac{m^2}{\mu^2} + \mathcal{O}(\epsilon)\right].
\end{aligned}
\tag{3.47}
$$

Since $-\gamma_E + \log(4\pi)$ always appear in the same combination with the $\frac{1}{\epsilon}$ term, and since they are numerically not small, it is customary to remove not only the $\frac{1}{\epsilon}$ terms when performing renormalization, but the entire combination. This is called the modified Minimal Subtraction (MS) scheme, or $\overline{\text{MS}}$ scheme. One way to achieve this, is to set $\mu^2 = \bar{\mu}^2 e^{\gamma_E}/(4\pi)$ so that

$$\Sigma^{(a)} = \frac{m^2 \lambda_L}{32\pi^2} \left[-\frac{1}{\epsilon} - 1 + \log \frac{m^2}{\bar{\mu}^2} + \mathcal{O}(\epsilon) \right]. \tag{3.48}$$

The calculation of diagram (b) is analogous to (a), leading to

$$\Sigma^{(b)} = \frac{M^2 \lambda_{HL}}{32\pi^2} \left[-\frac{1}{\epsilon} - 1 + \log \frac{M^2}{\bar{\mu}^2} + \mathcal{O}(\epsilon) \right]. \tag{3.49}$$

The expression for diagram (c) is the same as (a) apart from the additional heavy propagator

$$\Sigma^{(c)} = i \frac{1}{2} (-ig)^2 \mu^{2\epsilon} \frac{i}{-M^2} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2} \tag{3.50}$$

$$= \frac{g^2}{32\pi^2} \frac{m^2}{M^2} \left[\frac{1}{\epsilon} + 1 - \log \frac{m^2}{\bar{\mu}^2} + \mathcal{O}(\epsilon) \right]. \tag{3.51}$$

Diagram (d) is the only non-trivial contribution, namely

$$\Sigma^{(d)} = i(ig)^2 \int \frac{d^d k}{(2\pi)^d} \frac{i^2 \mu^{2\epsilon}}{[(k+p)^2 - m^2](k^2 - M^2)}. \tag{3.52}$$

We follow the strategy described in Appendix A to bring the integral into standard form, by introducing a Feynman parameter

$$\frac{1}{[(k+p)^2 - m^2](k^2 - M^2)} = \int_0^1 \frac{dx}{[(k+px)^2 + \underbrace{x(1-x)p^2 - xm^2 - (1-x)M^2}_{\equiv -\Delta(x)}]^2} \tag{3.53}$$

and shifting the integration variable by $k \to k - px$. This yields

$$\Sigma^{(d)} = -\frac{g^2 \mu^{2\epsilon}}{(4\pi)^{d/2}} \Gamma(\epsilon) \int_0^1 dx \, [\Delta(x)]^{-\epsilon}$$

$$= \frac{g^2}{16\pi^2} \left[-\frac{1}{\epsilon} + \log \frac{M^2}{\bar{\mu}^2} + \int_0^1 dx \log \frac{xm^2 + (1-x)M^2 - x(1-x)p^2}{M^2} \right]$$

$$= \frac{g^2}{16\pi^2} \left[-\frac{1}{\epsilon} + \log \frac{M^2}{\bar{\mu}^2} - 1 - \frac{p^2}{2M^2} - \frac{m^2}{M^2} \log \frac{m^2}{M^2} + \mathcal{O}\left(\frac{1}{M^4}\right) \right], \tag{3.54}$$

where in the last step we expanded in $1/M^2$. Note that the low-energy expansion does not commute with the integration. Naively expanding the integrand would lead to divergent integrals. This problem arises because near $x = 1$ the scale $(1-x)M^2$ becomes smaller than m^2 so that the expansion in m^2/M^2 is not justified. However, it is possible to expand in p^2 before the integration, which reduces the integral to

$$\int_0^1 dx \frac{-x(1-x)p^2}{xm^2 + (1-x)M^2} + \int_0^1 dx \log \frac{xm^2 + (1-x)M^2}{M^2} \tag{3.55}$$

$$= -\frac{p^2}{2M^2} - 1 + \frac{m^2 \log \frac{m^2}{M^2}}{m^2 - M^2} + \mathcal{O}\left(\frac{1}{M^4}\right). \tag{3.56}$$

Note that the result (3.54) for the integral contains a logarithm of m^2. This non-analytic dependence would not arise if one could Taylor expand the integrand.

3. Continuum effective theory

Effective-theory computation

The effective theory diagram reads

$$\Sigma = \frac{m^2 \tilde{\lambda}}{32\pi^2} \left[-\frac{1}{\epsilon} - 1 + \log \frac{m^2}{\bar{\mu}^2} \right] \tag{3.57}$$

and is obtained from the full theory diagram $\Sigma^{(a)}$ after substituting $\lambda_L \to \tilde{\lambda}$, the coupling in the effective theory.

Renormalization

We now renormalize, i.e., absorb the $\frac{1}{\epsilon}$-pieces of Σ into the couplings of the full and effective theory. To perform the renormalization, one would rewrite the all bare parameters in the Lagrangians in the form

$$\tilde{m}^2 = Z_m \, \tilde{m}^2(\mu) \,, \qquad\qquad \tilde{\lambda} = Z_\lambda \, \tilde{\lambda}(\mu) \,, \qquad\qquad \dots \tag{3.58}$$

where the Z-factors absorb the divergences arising in the loop diagrams. The quantities $\tilde{m}(\mu)$ and $\tilde{\lambda}(\mu)$ denote the renormalized mass and coupling, which depend on the scale μ, while the quantities without the μ argument refer to bare parameters. Note that the bare and renormalized quantities are the same, up to loop corrections since the Z-factors are equal to one at lowest order.

In addition, one also introduces a wave-function renormalization constant to absorb divergences in the normalization. For the effective theory, the Lagrangian then reads

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} Z_\phi (\partial_\mu \phi)^2 - \frac{\tilde{m}^2}{2} Z_\phi \phi^2 - \frac{\tilde{\lambda}}{4!} Z_\phi^2 \phi^4 \,. \tag{3.59}$$

We distinguish the field wave function renormalization Z_ϕ from the on-shell wave renormalization constant Z defined in (3.10). The the on-shell wave renormalization constant Z is the residue of the propagator pole, see (3.36), while the value of Z_ϕ depends on the choice of the renormalization scheme.

Looking at the two-point function in both the full and effective theory, we see that its divergence is p^2 independent. It can therefore be absorbed into mass renormalization

$$i\Gamma_2(p) = Z_\phi \, p^2 - Z_\phi Z_m \, \tilde{m}^2(\mu) - \Sigma \tag{3.60}$$

if we set $Z_\phi = 1$ and

$$Z_m = 1 + \frac{\tilde{\lambda}}{32\pi^2} \frac{1}{\epsilon} + \mathcal{O}(\tilde{\lambda}^2) \tag{3.61}$$

then the $i\Gamma_2(p)$ is finite up to the order of the computation. In exactly the same way, also the full theory two-point function can be renormalized. As discussed above the renormalization prescription which absorbs only the divergent terms after redefining $\mu \to \bar{\mu}$ is called the $\overline{\text{MS}}$ scheme.

Matching

After renormalization we now obtain a finite result for $\Gamma_2(p)$, both in the full and the effective theory. Next, we perform the matching, i.e. we adjust the the finite parts of the coupling constants in the effective theory so that they reproduce the full theory result. We note that diagram $i\Sigma^{(d)}$ in (3.54) involves a p^2-term. To reproduce this term, we will need to adjust the coefficient of the kinetic term in $\mathcal{L}_{\text{eff}}$ using a finite value for the wave-function renormalization factor Z_ϕ introduced above. To perform the matching we evaluate the difference between the two-point functions

$$\Delta = i\Gamma_2^{\text{eff}}(p) - i\Gamma_2^{\text{full}}(p), \tag{3.62}$$

which should be absorbed into the couplings of $\mathcal{L}_{\text{eff}}$. Explicitly, the matching condition reads

$$0 = \Delta = m^2 - p^2 - Z_\phi \tilde{m}^2 + Z_\phi \, p^2 + \Sigma_{\text{full}}^{(1\text{-loop})} - \Sigma_{\text{eff}}^{(1\text{-loop})}. \tag{3.63}$$

In order for this to work, all low-energy physics has to drop out of Δ. In particular, the $\log m^2$ pieces in Σ_{full} and Σ_{eff} have to cancel, since we cannot absorb a dependence on a low-energy scale into the Wilson coefficients. Let us look at these terms

$$\Delta = \frac{m^2}{32\pi^2} \log \frac{m^2}{\bar{\mu}^2} \left[\lambda_L - \frac{3g^2}{M^2} - \tilde{\lambda} \right] + \cdots , \tag{3.64}$$

where we used that $m = \tilde{m}$ up to higher orders in the coupling constants. In our tree-level calculation we found $\tilde{\lambda} = \lambda_L - \frac{3g^2}{M^2}$, so our $\mathcal{L}_{\text{eff}}$ indeed reproduces the low-energy part of the full theory.

$$\Delta = m^2(\mu) - Z_\phi \tilde{m}^2(\mu) + \frac{1}{16\pi^2} \left[g^2 \Big(1 + \frac{m^2}{M^2}\Big) + \frac{M^2}{2} \lambda_{HL} \right] \left(\log \frac{M^2}{\bar{\mu}^2} - 1 \right) - p^2 + Z_\phi \, p^2 - \frac{g^2}{32\pi^2} p^2. \tag{3.65}$$

where the mass renormalization has absorbed the divergences. In the terms suppressed by powers of the coupling constants we can identify $m = \tilde{m} = m(\mu) = \tilde{m}(\mu)$ up to higher orders. From the momentum-dependent term we read off

$$Z_\phi = 1 + \frac{g^2}{32\pi^2} , \tag{3.66}$$
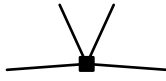
which amounts to a finite wave-function renormalization Z_ϕ. Note that we treat the normalization a bit different than the remaining parameters, where we have split the Wilson coefficients into Z-factor and a finite renormalized part, see (3.58).

$$\tilde{m}^2(\mu) = m^2(\mu)\Big(1 - \frac{g^2}{32\pi^2}\Big) + \frac{1}{16\pi^2} \left[g^2 \Big(1 + \frac{m^2}{M^2}\Big) + \frac{M^2}{2} \lambda_{HL} \right] \left(\log \frac{M^2}{\bar{\mu}^2} - 1 \right) + \cdots \tag{3.67}$$

We note the presence of a term proportional to $\lambda_{HL} M^2$. The same contributions will also arise in the physical mass m_{ph} determined by $\Gamma(p^2 = m_{\text{ph}}^2) = 0$. If m_{ph} is small, this implies a large cancellation among the terms in (3.67). This is again the statement that small scalar masses are unnatural.

Expanding the full theory result to higher powers $1/M^2$, one can determine the Wilson coefficients of power suppressed operators such as

$$-\frac{C_{4,2}}{4!M^2}\phi_L^2\Box\phi_L^2 \quad \equiv \qquad \tag{3.68}$$

In this case, additional one loop diagrams which involve the power suppressed operators will need to be computed in the effective theory. By computing the four- and six-point functions one can then determine also $\tilde{\lambda}$, $C_{4,2}$, and $C_{6,0}$ to one-loop accuracy. In this case, the number of diagrams becomes quite large. On the other hand, to perform the matching for $\tilde{\lambda}$, the four-point function at vanishing momentum is sufficient. We will come back to the matching for $\tilde{\lambda}$ below.

3.4. Power counting

In the matching calculation we have assumed that the higher-order Lagrangians do not contribute at leading power. At tree level, this is obvious, since the operators in the power-suppressed Lagrangians have additional derivatives and/or fields. But in loop integrals the momenta are large, so it is not immediately clear that higher-derivative terms are suppressed.

For instance, the contribution of $\frac{1}{(M^2)^n}\phi_L^2\Box^n\phi_L^2$ to the two-point function at zero external momentum reads

$$\delta\Sigma \propto \frac{1}{(M^2)^n}\int d^dk\,\frac{(k^2)^n}{k^2-m^2}, \tag{3.69}$$

and the loop integral, in principle, extends over all scales. The advantage of dimensional regularization is that the loop integrals in the EFT only depend on low-energy scales, therefore by dimensional analysis $\delta\Sigma \propto (m^2)^{d/2-1} \times (m^2)^n \times \frac{1}{(M^2)^n}$, since scaleless integrals are set to zero. Therefore, the loop contribution is indeed suppressed by $\left(\frac{m^2}{M^2}\right)^n$.

The power counting in dimensional regularization is thus very simple. One simply counts the $1/M$ suppression factors arising from the effective Lagrangian. If we compute a diagram with a single insertion of an irrelevant operator suppressed by $\left(\frac{1}{M^2}\right)^n$, then the contribution to the observable will be suppressed by $\left(\frac{1}{M^2}\right)^n$, independently of the loop order at which we compute with the leading-power effective Lagrangian. Similarly, if we compute a diagram involving two EFT operators, one suppressed by $\left(\frac{1}{M^2}\right)^n$ and one by $\left(\frac{1}{M^2}\right)^m$, then the total contribution is suppressed by $\left(\frac{1}{M^2}\right)^{n+m}$. Using dimensional regularization, it is therefore trivial to read off at which power in $1/M$ a given contribution enters.

Note that in a cutoff regularization

$$\frac{1}{(M^2)^n}\int^\Lambda d^dk\,\frac{(k^2)^n}{k^2-m^2} \propto \Lambda^{d-2} \times \frac{\Lambda^{2n}}{M^{2n}} + \cdots, \tag{3.70}$$

so that the loop contributions of higher-dimensional operators are not suppressed. The terms that violate the tree-level power counting are trivial cutoff terms and can thus be subtracted, but they make computations in a cutoff regularization cumbersome.

3.5. Renormalization-group improved perturbation theory

The Wilson coefficients C_i in $\mathcal{L}_{\text{eff}}$ depend on the coupling constants of the full theory as well as the large energy scale M. The dependence on M is logarithmic. For a full theory involving
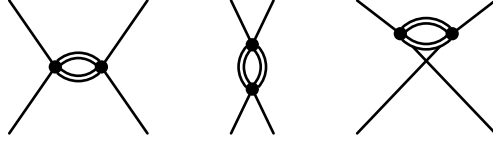
Figure 3.3.: One-loop corrections to the four-point function proportional to λ_{HL}^2.

a coupling constant λ and a large scale M, the result for the renormalized Wilson coefficients will have the form

$$C_i(\mu) = C_i^{(0,0)} + \lambda(\mu) \left[C_i^{(1,1)} \log \frac{M^2}{\mu^2} + C_i^{(1,0)} \right] \tag{3.71}$$

$$+ \lambda^2(\mu) \left[C_i^{(2,2)} \log^2 \frac{M^2}{\mu^2} + C_i^{(2,1)} \log \frac{M^2}{\mu^2} + C_i^{(2,0)} \right] + \cdots, \tag{3.72}$$

where the n-th order coefficients $C_i^{(n,m)}$, with $m \le n$, are pure numbers, determined by the matching calculation. The form of (3.71) makes it obvious that we should choose $\mu \approx M$, otherwise perturbation theory will not work well because the $\log \frac{M^2}{\mu^2}$ terms become large and will overwhelm the suppression by the coupling constant λ.

In our one-loop matching calculation for $\tilde{m}$ we found exactly this structure, in particular also the logarithmic dependence on M, except that our full theory has several different couplings instead of a single coupling λ. To better understand the consequences of the logarithmic scale dependence we will now study one-loop scattering in our scalar effective effective theory. For this we should extract the one-loop matching for the coupling $\tilde{\lambda}$. This involves a fairly large number of one-loop diagrams, but if we restrict ourselves to the contribution proportional to λ_{HL}^2, then only the three diagrams shown in Figure 3.3 contribute. To obtain the matching corrrection to $\tilde{\lambda}$, it is sufficient to evaluate these diagrams at zero external momentum. Together with the tree-level result (3.20), this gives the matching

$$\tilde{\lambda}(\mu) = \lambda(\mu) - \frac{3g^2(\mu)}{M^2} + \frac{3\lambda_{HL}^2(\mu)}{32\pi^2} \log \frac{M^2}{\mu^2} + \ldots \tag{3.73}$$

where the ellipsis denotes loop contributions involving the coupling g or λ_L. The couplings $\tilde{\lambda}(\mu)$, $\lambda_{HL}(\mu)$ and $g(\mu)$ are all renormalized in the $\overline{\text{MS}}$ scheme and for simplicity we write $\bar{\mu} \to \mu$ from now on. To obtain a reliable result for $\tilde{\lambda}(\mu)$, the coupling $\lambda_{HL}(\mu)$ must be small and $\mu \approx M$ so that the logarithm is not too large.

The logarithmic dependence on the renormalization scale μ is quite characteristic for higher-order corrections in quantum field theory and also arises in the effective theory. To see this, consider the $2 \to 2$ amplitude at next-to-leading order, for which we get

$$\mathcal{M} = \vcenter{\hbox{}}$$

$$= -\tilde{\lambda} \left[1 + \frac{3\tilde{\lambda}}{32\pi^2} \left(-\frac{1}{\epsilon} + \log \frac{m^2}{\mu^2} + f(s,t,u) \right) \right]$$

$$= -\tilde{\lambda}(\mu) \left[1 + \frac{3\tilde{\lambda}(\mu)}{32\pi^2} \left(\log \frac{m^2}{\mu^2} + f(s,t,u) \right) \right], \tag{3.74}$$

3. Continuum effective theory

The finite part of the amplitude can be written as

$$f(s,t,u) = V(s) + V(t) + V(u), \qquad V(s) = \frac{1}{3} \int_0^1 dx \log \frac{m^2 - x(1-x)s}{m^2}, \qquad (3.75)$$

with the Mandelstam variables

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2. \qquad (3.76)$$

Its precise form is irrelevant for the following discussion, but we see that $V(s)$ is a number of order 1 as long as $s \sim m^2$ as is characteristic for low-energy scattering. However, we again have logarithmic dependence, this time of the form $\log \frac{m^2}{\mu^2}$ so that we should choose $\mu \approx m$ to keep the logarithm small. We see that this leads to conflicting requirements: (i) the matching in (3.73) requires $\mu \approx M$, while (ii) the EFT matrix element (3.74) requires $\mu \approx m$, but our starting point was $m \ll M$ so we cannot fulfill both conditions (i) and (ii) simultaneously. The large logarithms are not specific to the effective theory, but are also present in the full theory as terms of the form $\lambda^n \log^n \frac{m^2}{M^2}$ and this results in a breakdown of perturbation theory (in the $\overline{\text{MS}}$ scheme) for $m \ll M$, even if λ is very small. In the presence of large scale hierarchies in a quantum field theory, perturbative calculations become unreliable since the higher-order corrections are enhanced by large logarithms.

Fortunately, the renormalization group in the EFT allows us to resum the logarithmically enhanced terms to all orders by solving renormaliztion evolution equations. For a single coupling constant, the renormaliztion group equation is written in the form

$$\frac{d\tilde{\lambda}(\mu)}{d \log \mu} = \mu \frac{d\tilde{\lambda}(\mu)}{d\mu} = \beta(\tilde{\lambda}(\mu)). \qquad (3.77)$$

The β-function on the right determines the evolution of the renormalized coupling as a function of the renormalization scale μ. For general Wilson coefficients, the evolution equations are written as a matrix equation

$$\frac{dC_i(\mu)}{d \log \mu} = C_j(\mu) \, \gamma_{ji}(\tilde{\lambda}(\mu)). \qquad (3.78)$$

Operators of the same dimension "mix", i.e., their RG evolution is coupled and the scale evolution is driven by the anomalous dimension matrix $\gamma_{ij}(\tilde{\lambda}(\mu))$. The fact that (3.77) and (3.78) are written in different form is historical, we could extract a factor $\tilde{\lambda}(\mu)$ from $\beta(\tilde{\lambda}(\mu))$ and write it in the form (3.78).

The strategy to resum large logarithms using renormalization group equations in the effective theory is illustrated in Fig. 3.5. One can first match at a high scale $\mu_h = M$ which avoids large logarithms in the matching computations. By solving the RG evolution equations, one then evolves the Wilson coefficients to a lower scale. Performing the EFT computations at the low scale $\mu_l = m$ then avoids large logarithms in the EFT matrix elements. Solving the RG evolution equation resums the logarithically enhanced corrections to all orders. Let us illustrate this procedure for the leading-order four-point function. To determine the β-function, we use

$$\frac{d}{d \log \mu} \mathcal{M} = 0, \qquad (3.79)$$
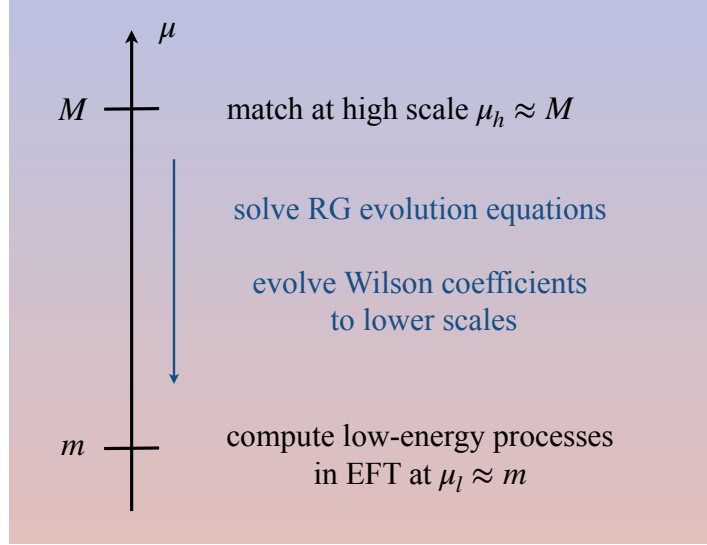
Figure 3.4.: General strategy for computations in EFT. By performing the matching at a high scale $\mu = M$ one avoids large logarithms. After solving the associated RG evolution equations, one can choose $\mu = m$ to avoid large logarithms in the EFT matrix elements.

since the physical amplitude is μ-independent. This gives

$$\frac{d}{d\log(\mu)}\tilde{\lambda}(\mu) = \beta(\tilde{\lambda}) = \frac{3\tilde{\lambda}^2(\mu)}{16\pi^2} + \cdots , \tag{3.80}$$

We now solve this equation to evolve from the high scale $\mu_h \approx M$ to the low scale $\mu_l \approx m$. The equation can be solved via a separation of variables

$$\int_{\tilde{\lambda}(\mu_h)}^{\tilde{\lambda}(\mu_l)} \frac{d\tilde{\lambda}}{\tilde{\lambda}^2} = \frac{3}{32\pi^2} \log \frac{\mu_l^2}{\mu_h^2}, \tag{3.81}$$

yielding

$$\frac{1}{\tilde{\lambda}(\mu_h)} - \frac{1}{\tilde{\lambda}(\mu_l)} = \frac{3}{32\pi^2} \log \frac{\mu_l^2}{\mu_h^2}. \tag{3.82}$$

The solution takes the form

$$\tilde{\lambda}(\mu_l) = \frac{\tilde{\lambda}(\mu_h)}{1 - \frac{3}{16\pi^2}\tilde{\lambda}(\mu_h)\log\frac{\mu_l}{\mu_h}} . \tag{3.83}$$

The coupling at the low scale $\mu \approx m$ is then used to compute the scattering amplitude

$$\mathcal{M} = -\tilde{\lambda}(\mu_l)\left[1 + \frac{3\tilde{\lambda}(\mu_l)}{32\pi^2}\left(\log\frac{m^2}{\mu_l^2} + f(s,t,u)\right)\right] . \tag{3.84}$$

By performing the matching in (3.73) at a high scale $\mu_h \approx M$ and inserting the result into (3.83) to get the coupling at the low scale μ_l, where we evaluate the scattering amplitude, we have successfully avoided large logarithms in the computation.

Looking at (3.83), we see that it contains terms of the form $\tilde{\lambda}(\mu_h) \log \frac{\mu_l}{\mu_h}$ to all orders in perturbation theory. Indeed, (3.82) shows that we count the large logarithm as $1/\tilde{\lambda}$. The procedure we follow is called RG improved perturbation theory. It eliminates the large logarithms in favor of coupling constants at the high and low scale, see (3.82). One then expands in both couplings which are counted as of the same order.

At leading order in RG-improved perturbation theory, one resums the logarithms $\tilde{\lambda}^n \log^n \frac{\mu_h^2}{\mu_l^2}$ by solving the leading order (i.e. one-loop) RG equation for the Wilson coefficients, together with matching at the leading order. This procedure drops all terms suppressed by additional factors of $\tilde{\lambda}(\mu_h)$ or $\tilde{\lambda}(\mu_l)$. At leading order, one would thus drop the one-loop corrections both in (3.73) and (3.84).

At next-to-leading order in RG-improved perturbation theory, one needs to include the one-loop corrections in both (3.73) and (3.84) and take into account also the corrections to the RG evolution, which are of the form $\tilde{\lambda} \times \tilde{\lambda}^n \log^n \frac{\mu_h^2}{\mu_l^2} \sim \tilde{\lambda}$. This is achieved by solving the evolution with the two-loop β-function:

$$\frac{d\tilde{\lambda}(\mu)}{d\log\mu} = \beta(\tilde{\lambda}(\mu)) = \tilde{\lambda}(\mu)\left[3\frac{\tilde{\lambda}(\mu)}{16\pi^2} - \frac{17}{3}\left(\frac{\tilde{\lambda}(\mu)}{16\pi^2}\right)^2 + \cdots\right], \qquad (3.85)$$

so that

$$\frac{3}{16\pi^2}\log\frac{\mu_l}{\mu_h} = \frac{1}{\tilde{\lambda}(\mu_h)} - \frac{1}{\tilde{\lambda}(\mu_l)} + \frac{17}{9}\times\frac{1}{16\pi^2}\log\frac{\tilde{\lambda}(\mu_l)}{\tilde{\lambda}(\mu_h)} + \mathcal{O}(\tilde{\lambda}). \qquad (3.86)$$

The phenomenon we illustrated with out toy model is a very general one: In the presence of disparate scales, large logarithms of scale ratios destroy the perturbative expansion. The use of an EFT allows one to disentangle the different scales and, by using RG evolution, resum the logarithmically enhanced contributions. The prime example of a theory involving disparate scales is the SM, which involves particles with very different masses. To avoid large logarithms, low-energy calculations are never performed using the SM Lagrangian directly, but an effective Lagrangian obtained by "integrating out" heavy particles such as t-quarks, Higgs, $W^\pm$, Z^0.

3.6. Summary

Having worked out the construction of an EFT in our scalar-field toy example, we are now ready for real-life physics applications of this technology. However, before discussing EFTs within the context of the Standard Model, let us recapitulate the individual steps in the construction and of the effective theory:

1. Identify the degrees of freedom at low energy.

2. Construct the most general $\mathcal{L}$ with these degrees of freedom and the symmetries of the full theory.

   a) Higher-dimensional operators in $\mathcal{L}_{\text{eff}}$ are suppressed by $(\frac{1}{M})^n$, where M is a characteristic high-energy scale. Their contribution to observables is suppressed by $(\frac{E}{M})^n$, so only a finite number of terms is needed for a given accuracy ϵ: $n \approx \frac{\log\epsilon}{\log(E/M)}$.

   b) Field redefinitions: higher-order terms in $\mathcal{L}_{\text{eff}}$ that vanish by the leading-order EOM do not contribute to physical amplitudes and can be omitted from $\mathcal{L}_{\text{eff}}$.

3. Matching: determine the Wilson coefficients of the operators in $\mathcal{L}_{\text{eff}}$ by computing a number of quantities in both the full and the effective theory. Adjust the couplings in $\mathcal{L}_{\text{eff}}$ to reproduce the full theory result. If field redefinitions have been used, only physical quantities match, otherwise arbitrary Green's functions can be reproduced.

4. RG improvement: compute the anomalous dimensions and solve the RG equations for the operators in $\mathcal{L}_{\text{eff}}$, i.e.,

$$C_i(\mu) = \sum_j \underbrace{U_{ij}(\mu_h, \mu)}_{\substack{\text{Evolution} \\ \text{from } \mu_h \approx M \\ \text{to } \mu \approx E \\ \text{resums logarithms}}} \underbrace{C_j(\mu_h)}_{\substack{\text{no large logarithms} \\ \text{for } \mu_h \approx M}} . \tag{3.87}$$

4. The Standard Model at low energies

In this chapter, we turn to low-energy effective theories for the interactions of the Standard Model (SM). In each case, the EFT approach simplifies calculations for a given sector of the SM and is sometimes the only viable strategy to perform analytic calculations. We will first treat electromagnetism (Euler–Heisenberg theory), before turning to weak and strong interactions.

4.1. Euler–Heisenberg theory

Let us consider the QED Lagrangian

$$\mathcal{L}_{\text{QED}}[A_\mu, \psi] = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\slashed{D} - m)\psi, \tag{4.1}$$

where

$$iD_\mu = i\partial_\mu - eA_\mu, \tag{4.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{e}[iD_\mu, iD_\nu]. \tag{4.3}$$

A_μ is the electromagnetic potential and ψ the electron field. Note that $\mathcal{L}_{\text{QED}}$ is the most general renormalizable Lagrangian for an electron interacting with the photon field. In the SM, there are many other heavier charged particles (quarks, $W^\pm$, μ, τ), but according to EFT logic, the contributions of all the heavier fields only appear via $\frac{1}{M}$ suppressed operators (where $M = m_\tau, M_\pi, \dots$). In other words, $\mathcal{L}_{\text{QED}}$ is the leading-order effective Lagrangian describing the interactions of $e^\pm$ and γ, and it will be appropriate as long as $E \ll 100\,\text{MeV} \approx m_\mu \sim M_\pi$.

Many practical applications only involve photons at even smaller energies $E \ll 2m_e$. In this case, electron-positron pairs appear only as virtual corrections and we can integrate them out, i.e., construct an effective theory involving only photons. To do so, we first encounter an interesting complication: since electron number is conserved, a given state, say with $5e^-$, will be there even for $E \to 0$. To describe such a situation correctly, one has to use non-relativistic EFT which we will cover in a later chapter. For now, we concentrate on the sector with zero net electrons, in which $e^\pm$ only appear as virtual particles. To do so, we can describe the electrons as an external current and add a term

$$\mathcal{L}_J = -eA_\mu J^\mu \tag{4.4}$$

to the Lagrangian. This description should work for macroscopic charged objects, as long as we do not excite higher energy levels in their interaction with the photon. Note that this interaction is only consistent if $\partial_\mu J^\mu = 0$. Under gauge transformations

$$A^\mu \to A^\mu + \partial^\mu\phi, \tag{4.5}$$

so

$$\int d^4x\, A_\mu J^\mu \to \int d^4x A_\mu J^\mu - \int d^4x \phi \underbrace{\partial_\mu J^\mu}_{=0}. \qquad (4.6)$$

Examples of configurations fulfilling $\partial_\mu J^\mu = 0$ are

1. static charge distribution: $J^\mu = (\rho(r), \mathbf{0})$

2. static current: $J^\mu = (0, \mathbf{j}(r))$ with $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$

If we now consider low-energy photons in the background of a source J^μ, we should be able to describe their interactions with the effective Lagrangian

$$\mathcal{L}_{\text{eff}}[A_\mu, J_\mu] = \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \mathcal{L}^{(8)} + \dots \qquad (4.7)$$

The leading-order Lagrangian is

$$\mathcal{L}^{(4)} = -\frac{Z}{4} F^{\mu\nu} F_{\mu\nu} - e A_\mu J^\mu \qquad (4.8)$$

and describes free photons. Let us now construct the operators of dimension 6 and 8, whose effects are suppressed by $\mathcal{O}(m_e^{-2})$ and $\mathcal{O}(m_e^{-4})$, respectively. To obtain $\mathcal{L}_{\text{eff}}$, one writes down all possible terms of a given dimension. The number of terms can be reduced to a minimal set using

(i) Symmetries, e.g., charge conjugation. QED is invariant under

$$e \to -e, \qquad A_\mu \to -A_\mu, \qquad F_{\mu\nu} \to -F_{\mu\nu},$$

and so has to be the effective theory.

(ii) Properties of $F_{\mu\nu}$, e.g., its antisymmetry $F^{\mu\nu} = -F^{\nu\mu}$ and the Bianchi identity

$$\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0.$$

(iii) The leading-order EOM $\partial_\mu F^{\mu\nu} = J^\nu$.

We start with the $d = 6$ terms. Because of charge conjugation symmetry, $\mathcal{L}_{\text{eff}}$ must be even in $F^{\mu\nu}$. This is the EFT equivalent of Furry's theorem, which states that amplitudes with an odd number of photons vanish in QED.

$$= 0 \qquad (4.9)$$

This leaves us with terms of the form $\partial^2 F^2$. Using integration by parts, we can always achieve

that derivatives are not contracted with the field strength on which they act. This leaves two possible terms:

$$O_1 = F^{\mu\nu}\Box F_{\mu\nu}, \tag{4.10}$$

$$O_2 = (\partial^\rho F^{\mu\nu})(\partial_\mu F_{\rho\nu}). \tag{4.11}$$

Using the Bianchi identity on O_2, we find

$$O_2 = (\partial^\rho F^{\mu\nu})\left[-\partial_\rho F_{\nu\mu} - \partial_\nu F_{\mu\rho}\right] \tag{4.12}$$

$$= F^{\mu\nu}\Box F_{\nu\mu} - \partial^\rho F^{\mu\nu}\partial_\nu F_{\mu\rho} \; + \; \text{total derivative} \tag{4.13}$$

$$= -F^{\mu\nu}\Box F_{\mu\nu} - \partial^\rho F^{\nu\mu}\partial_\nu F_{\rho\mu} \; + \; \text{total derivative} \tag{4.14}$$

$$= -O_1 - O_2 \; + \; \text{total derivative}, \tag{4.15}$$

so that the terms O_1 and O_2 are equivalent, $2O_2 \hat{=} -O_1$. In addition, we can write down terms

$$O_3 = J_\mu J^\mu, \qquad O_4 = \partial_\mu F^{\mu\nu} J_\nu, \tag{4.16}$$

since J^μ has dimension $d = 3$. These two terms are equivalent upon using the EOM $\partial_\mu F^{\mu\nu} = J^\nu$. Moreover, up to total derivatives also O_2 can be brought into the form $O_2 \hat{=} \partial_\mu F^{\mu\nu}\partial^\rho F_{\rho\nu}$, in such a way that all operators become equivalent to O_3. Our final result can therefore be expressed as

$$\mathcal{L}^{(6)} = \frac{C_0}{m_e^2} J^\mu J_\mu, \tag{4.17}$$

which corresponds to a contact interaction between the source and is irrelevant for photon propagation or scattering.

The first terms involving photons appear for $d = 8$

$$\mathcal{L}^{(8)} = \frac{C_1}{m_e^4}\left(F^{\mu\nu}F_{\mu\nu}\right)^2 + \frac{C_2}{m_e^4}F_{\mu\nu}F^{\nu\sigma}F_{\sigma\rho}F^{\rho\mu}. \tag{4.18}$$

In four space-time dimensions, we can rewrite

$$F_{\mu\nu}F^{\nu\sigma}F_{\sigma\rho}F^{\rho\mu} = \frac{1}{4}(F^{\mu\nu}\tilde{F}_{\mu\nu})^2 + \frac{1}{2}\left(F^{\mu\nu}F_{\mu\nu}\right)^2, \tag{4.19}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. However, since $\epsilon^{\mu\nu\rho\sigma}$ is only defined in $d = 4$, it is preferable not to use this relation.[1] Expressed in terms of $\mathbf{E}$ and $\mathbf{B}$ the two structures are

$$F^{\mu\nu}F_{\mu\nu} = -2\left(\mathbf{E}^2 - \mathbf{B}^2\right), \qquad \tilde{F}^{\mu\nu}F_{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}. \tag{4.21}$$

The two terms in $\mathcal{L}^{(8)}$ describe four-point interactions, but since they are suppressed by $\mathcal{O}(m_e^{-4})$, they will be very weak at low energies where the EFT applies. In QED, these interactions arise from fermion loops

---

[1]To derive it, use:

$$\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\nu_1\nu_2\nu_3\nu_4} = -\begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \cdots \\ \vdots & & \\ \delta^{\nu_1}_{\mu_4} & \cdots & \delta^{\nu_4}_{\mu_4} \end{vmatrix} \tag{4.20}$$
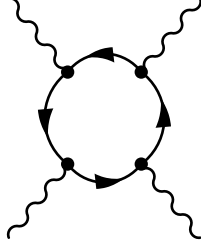
Before considering these diagrams, let us study the low-energy $\gamma\gamma \to \gamma\gamma$ scattering cross section. This cross section scales according to

$$d\sigma \sim \left(\frac{1}{m_e^4}\right)^2 E^6 e^8, \qquad e^8 \propto \alpha^4. \tag{4.22}$$

The factor E^6 is required to get the correct dimension $d\sigma \sim E^{-2}$. An explicit computation yields the unpolarized cross section:

$$\frac{d\sigma_{\gamma\gamma}}{d\Omega} = \frac{1}{4\pi^2}(48C_1^2 + 40C_1C_2 + 11C_2^2)\frac{E^6}{m_e^8} \times (3 + \cos^2\theta)^2, \tag{4.23}$$

where $\cos\theta$ is the scattering angle in the center-of-mass system. So far, $\gamma\gamma$ scattering for $E < m_e$ has not been observed experimentally, but there are plans to measure it using intense lasers [10].[2]

To determine the Wilson coefficients C_1 and C_2, we need to perform a matching computation. It is simplest to consider the $\gamma\gamma$ scattering amplitude directly. Since we only need to extract two numbers, it suffices to evaluate the forward amplitude $\gamma(p_1)+\gamma(p_2) \to \gamma(p_1)+\gamma(p_2)$ and to consider two different helicity configurations. In QED, the amplitude is given by a sum of box diagrams. Let us define the basic box diagram as follows

$$\mathcal{B}^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4) = \qquad\qquad\qquad\qquad\qquad\qquad \tag{4.24}$$

The full amplitude involves six box diagrams, which can be obtained from $\mathcal{B}$ by exchanging external legs. The exchange $2 \leftrightarrow 3$ is equivalent to $1 \leftrightarrow 4$ and corresponds to reversing the direction of the fermion line in $\mathcal{B}$ from $\circlearrowleft$ to $\circlearrowright$. This yields an identical result to $\mathcal{B}$ and leads to a factor two in the scattering amplitude. Exchanging the adjacent legs $3 \leftrightarrow 4$ and $2 \leftrightarrow 4$ yields two additional in-equivalent diagrams, each of which has a counterpart with a

---

[2]At higher energies, $\gamma\gamma$ scattering has been seen in heavy-ion collisions [11, 12].

reversed Feynman line. Taking into account a minus sign if initial- and final-state particles are exchanged, the full amplitude is thus obtained by evaluating

$$
\begin{aligned}
\mathcal{A}^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4) =& 2\mathcal{B}^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4) + 2\mathcal{B}^{\mu_1\mu_2\mu_4\mu_3}(p_1, p_2, p_4, p_3) \\
&+ 2\mathcal{B}^{\mu_1\mu_4\mu_3\mu_2}(p_1, -p_4, p_3, -p_2) \, .
\end{aligned}
\tag{4.25}
$$

and contracting the result with polarization vectors for the incoming and outgoing photons

$$
\mathcal{M}(p_1, p_2, p_3, p_4) = \mathcal{A}^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4)\varepsilon_{\mu_1}\varepsilon_{\mu_2}\varepsilon^*_{\mu_3}\varepsilon^*_{\mu_4} .
\tag{4.26}
$$

Each photon comes in two polarizations λ, i.e., $\varepsilon_{\mu_i} \equiv \varepsilon_{\mu_i}(\lambda_i)$, but we suppress the polarization dependence. To perform the matching we do not need the full information on the scattering amplitude, only enough to determine the two coefficients C_1 and C_2. For our purpose, the forward scattering $p_3 = p_1$ $p_4 = p_2$ is sufficient and we only need to spin structures, for instance

$$
\mathcal{A}_1 = g^{\mu_1\mu_2}g^{\mu_3\mu_4}\mathcal{A}_{\mu_1\mu_2\mu_3\mu_4}, \qquad \mathcal{A}_2 = g^{\mu_1\mu_3}g^{\mu_2\mu_4}\mathcal{A}_{\mu_1\mu_2\mu_3\nu_4},
\tag{4.27}
$$

in both the full and the effective theory and then solve for C_1, C_2. The computation can be further simplified by expanding the QED diagrams in the small external momenta, which can be done on the integrand level in this case. After the expansion, the necessary integrals all have the form

$$
\int d^d k \frac{(k^2)^\alpha}{(k^2 - m_e^2)^\beta}
$$

and are obtained directly from Appendix A, leading to

$$
C_1 = -\frac{1}{36}\alpha^2, \qquad C_2 = \frac{7}{90}\alpha^2.
\tag{4.28}
$$

In particular, the result for the one-loop amplitude is finite and does not contain any logarithmic corrections proportional to $\log \frac{\mu^2}{m_e^2}$. In the EFT we can directly understand the reason for this cancellation: the only loop in the EFT from $\mathcal{L}^{(8)}$ is

$$
= 0,
\tag{4.29}
$$

since the photon loop only contains scaleless integrals. Other loop diagrams are not possible because a second vertex from $\mathcal{L}^{(8)}$ would give an additional m_e^{-4} suppression and $\mathcal{L}^{(4)}$ and $\mathcal{L}^{(6)}$ do not contain interactions. The operators in $\mathcal{L}^{(8)}$ are thus not renormalized. Plugging in C_1 and C_2 into our earlier result for the cross section, one has

$$
\frac{d\sigma_{\gamma\gamma}}{d\Omega} = 139\left(\frac{\alpha^2}{180\pi}\right)^2 \left(3 + \cos^2\theta\right)^2 \frac{E^6}{m_e^8} .
\tag{4.30}
$$

4.2. Decoupling of heavy flavors

The quarks and leptons in the SM appear in three generations. For reasons we do not understand, the masses of the fermions are quite hierarchical $m_1 \ll m_2 \ll m_3$, e.g., $m_e = 0.5$ MeV, $m_\mu = 106$ MeV, $m_\tau = 1777$ MeV. An important generalization of the Euler–Heisenberg EFT is the EFT obtained from integrating out heavy flavors. For QED at different energies, one uses

$$
\begin{aligned}
E \gtrsim m_\tau \qquad & \mathcal{L}_{\mathrm{QED}}[\tau, \mu, e, A] \\
& \downarrow \\
m_\tau \gtrsim E \gtrsim m_\mu \qquad & \mathcal{L}_{\mathrm{QED}}^{\mathrm{eff}}[\mu, e, A] \\
& \downarrow \\
m_\mu \gtrsim E \gtrsim m_e \qquad & \mathcal{L}_{\mathrm{QED}}^{\mathrm{eff}}[e, A] \\
& \downarrow \\
m_e \gtrsim E \qquad & \mathcal{L}_{\mathrm{Euler–Heisenberg}}[A]
\end{aligned}
$$

Since $M_\pi \sim m_\mu$, one needs also to consider strong interaction effects once one includes the muon in the Lagrangian, but we will ignore this complication for the moment and start with $\mathcal{L}_{\mathrm{QED}}^{\mathrm{eff}}[\mu, e, A]$ and construct $\mathcal{L}_{\mathrm{QED}}^{\mathrm{eff}}[e, A]$. We will then discuss how this Lagrangian can be used to search for physics beyond the SM and how the analog construction works in the QCD case.

4.2.1. Heavy flavors in QED

The leading-order Lagrangian $\mathcal{L}_{\mathrm{QED}}^{\mathrm{eff}}[e, A]$ is just the QED Lagrangian

$$
\mathcal{L}_{\mathrm{QED}} = \bar{\psi}(i\slashed{D} - m_e)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.31}
$$

which contains the two parameters e and m_e to be determined by matching. At higher orders, we get the same photonic operators as in the Euler–Heisenberg case. In addition, there are now operators containing fermion fields. Up to operator dimension $d = 6$, we have

$$
\bar{\psi}\Gamma^\mu D_\mu\psi, \quad \bar{\psi}\Gamma^{\mu\nu} D_\mu D_\nu\psi, \quad \bar{\psi}\Gamma^{\mu\nu\rho} D_\mu D_\nu D_\rho\psi, \quad \bar{\psi}\Gamma_1\psi\bar{\psi}\Gamma_2\psi, \tag{4.32}
$$

where the Γ are arbitrary Dirac matrices. At $d = 4$, the only possibility is $\bar{\psi}i\slashed{D}\psi$. At $d = 5$ one has

$$
O_1 = \bar{\psi}\frac{1}{2}[\gamma^\mu, \gamma^\nu]D_\mu D_\nu\psi = \bar{\psi}(-i\sigma^{\mu\nu})\frac{1}{2}[D_\mu, D_\nu]\psi \tag{4.33}
$$

$$
= \frac{e}{2}\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi, \tag{4.34}
$$

$$
O_2 = \bar{\psi}\underbrace{\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}}_{g_{\mu\nu}} D_\mu D_\nu\psi, \tag{4.35}
$$

where we used Eq. (4.3) for the covariant derivatives. $O_1 + O_2 = \bar{\psi}\slashed{D}\slashed{D}\psi$ can be eliminated using the quark-field EOM $i\slashed{D}\psi = m\psi$, so we only need to consider one operator, e.g., O_1. It turns out that the Wilson coefficients of O_1 vanishes for $m_e = 0$. The reason is that $\mathcal{L}_{\mathrm{QED}}$ has a symmetry $\psi \to e^{i\alpha\gamma_5}\psi$, $\bar{\psi} \to \bar{\psi}e^{+i\alpha\gamma_5}$ for $m_e = 0$ and O_1 violates this symmetry. So we

only need to consider the $d = 6$ operator $O_{\text{mag}} = m_e O_1 = m_e \bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi$. As a side remark, we note that the axial symmetry $\psi \to e^{i\alpha\gamma_5}\psi$ is *not* a symmetry of the theory, but only of the Lagrangian because the path-integral measure is not invariant. However, the measure for ψ is the same in the full and effective theory, so the axial anomaly does not affect our argument.

At d=6, we have operators with three covariant derivatives on the fermions as well as the four-fermion operators

$$O_{(n)} = \bar{\psi}\Gamma_{(n)}\psi\bar{\psi}\Gamma_{(n)}\psi \tag{4.36}$$

with $\Gamma_{(n)} = \gamma^{[\alpha_1}\gamma^{\alpha_2}\ldots\gamma^{\alpha_n]}$ (totally antisymmetrized) at $d = 6$. Symmetric products are sufficient because symmetric products can be reduced using the Dirac algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu,\nu}$. Because of axial symmetry, the terms with even n will have coefficients $\propto m_e$ and contribute at the same level as $d = 7$ operators. In $d = 4$, there a are only 16 independent Dirac matrices and products of more than four Dirac matrices can always be reduced. However, this is not true in dimensional regularization where d is kept arbitrary. The extra operators not present in four dimensions are called evanescent operators. While their matrix elements vanish in $d = 4$, they need to be included to renormalize the theory consistently, see e.g. [9]. Here, we will not discuss this point further. The operators with three covariant derivatives all reduce to O_{mag} and $O_{(1)}$ using the leading-power EOMs

$$i\slashed{D}\psi = m\psi \tag{4.37}$$
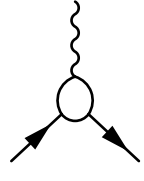
$$\partial_\mu F_{\mu\nu} = e\bar{\psi}\gamma^\nu\psi \tag{4.38}$$

Note in particular that

$$\bar{\psi}\partial_\sigma F_{\mu\nu}\gamma^{[\mu}\gamma^\nu\gamma^{\sigma]}\psi = 0 \tag{4.39}$$

from the Bianchi identity. Accordingly, we conclude that, up to terms suppressed by at least m_μ^{-3}, all effects of physics at scales $E \ll m_\mu$ can be absorbed into the electron mass and the electromagnetic coupling, as well as the Wilson coefficients of O_{mag}, $O_{(1)}$, and $O_{(3)}$.

Let us now discuss the physics associated with O_{mag} and how the effects of physics beyond the Standard Model manifest themselves in C_{mag}. For this purpose, we consider the interaction of an electron with a background electromagnetic field

$$= \bar{u}(p_2)\Gamma^\mu(p_1, p_2)u(p_1) \times (-ieA_\mu), \tag{4.40}$$

p_1

where the independent Dirac structures are

$$\Gamma^\mu = A\gamma^\mu + B(p_1 + p_2)^\mu + C(p_1 - p_2)^\mu + A'\gamma^\mu\gamma^5 + B'(p_1 + p_2)^\mu\gamma^5 + C'(p_1 - p_2)^\mu\gamma^5. \tag{4.41}$$

The coefficients A', B', C' are zero because of parity invariance of QED. Additional structures involving $\slashed{p}_1$ or $\slashed{p}_2$ can be eliminated using the EOM $\slashed{p}u(p) = m_e u(p)$. The coefficients are scalar functions and depend on q^2, where $q = p_2 - p_1$, and $p_1^2 = p_2^2 = m_e^2$. Furthermore, current conservation implies

$$0 = \langle e(p_2)| \, i\partial_\mu j^\mu \, |e(p_1)\rangle = q^\mu\bar{u}(p_2)\Gamma_\mu u(p_1) = Cq^2 \Rightarrow C = 0. \tag{4.42}$$

We are thus left with two functions $A(q^2)$ and $B(q^2)$. It is customary to write Γ^μ in the form

$$\Gamma^\mu = F_1(q^2)\gamma^\mu + \frac{i}{2m_e}\sigma^{\mu\nu}q_\nu F_2(q^2)\,. \tag{4.43}$$

At tree level in the EFT, we have $F_1(q^2) = 1$ and $F_2(q^2) = C_{\mathrm{mag}}m_e^2$, where we wrote

$$\mathcal{L}_{\mathrm{mag}} = \frac{C_{\mathrm{mag}}}{4}O_{\mathrm{mag}} = \frac{C_{\mathrm{mag}}}{4}m_e\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi\,. \tag{4.44}$$

In QED $F_1 = 1$ at tree level, while F_2 is only generated by loop contributions. To understand the meaning of F_1 and F_2, let us consider the non-relativistic limit $\mathbf{p}_1$, $\mathbf{p}_2 \to 0$. In the basis

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}\,, \qquad\qquad \gamma^i = \begin{pmatrix} & \sigma^i \\ \sigma^i & \end{pmatrix}\,, \tag{4.45}$$

one has

$$u(p) = \sqrt{p^0 + m_e}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m_e}\chi_s \end{pmatrix} = \sqrt{2m_e}\begin{pmatrix} \chi_s \\ 0 \end{pmatrix} + \mathcal{O}(\mathbf{p})\,. \tag{4.46}$$

Moreover, using the Gordon identity

$$\bar{u}(p_2)\gamma^\mu u(p_1) = \bar{u}(p_2)\left[\frac{(p_1+p_2)^\mu}{2m_e} + \frac{i}{2m_e}\sigma^{\mu\nu}q_\nu\right]u(p_1) \tag{4.47}$$

we can replace the γ^μ term in favor of $(p_1+p_2)^\mu$, which is easier to handle in the non-relativistic expansion

$$A_\mu\frac{(p_1+p_2)^\mu}{2m_e}\bar{u}(p_2)u(p_1) = 2m_e A^0 \chi_s^\dagger\chi_s + \mathcal{O}(\mathbf{p}^2), \tag{4.48}$$

and

$$\begin{aligned} A_\mu\bar{u}(p_2)\frac{i}{2m_e}\sigma^{\mu\nu}q_\nu u(p_1) &\cong A_i\bar{u}(p_2)\frac{-1}{4m_e}[\gamma^i,\gamma^j]q_j u(p_1) \\ &\cong -\frac{A_i}{4m_e}2m_e\chi_s^\dagger\sigma^k\chi_s 2i\epsilon^{ijk}q_j \\ &= -iA_i q_j\chi_s^\dagger\sigma_k\chi_s\epsilon^{ijk} \\ &= -\chi_s^\dagger\boldsymbol{\sigma}\cdot\mathbf{B}(q)\chi_s, \end{aligned} \tag{4.49}$$

where $B_k(q) = -i\epsilon^{ijk}q_iA_j(q)\hat{=}(\boldsymbol{\nabla}\times\mathbf{A}(q))_k$ and we used $[\sigma^i,\sigma^j] = 2i\epsilon^{ijk}\sigma^k$. In the first line, we dropped $[\gamma^i,\gamma^0]$ since it mixes the upper and lower component of the spinors and is thus suppressed by $\mathcal{O}(p)$ compared to $[\gamma^i,\gamma^j]$. The QM Hamiltonian describing the interaction of an electron with an electromagnetic field contains a term

$$\mathbb{H} = -g_e\frac{e}{2m_e}\mathbf{S}\cdot\mathbf{B} = -\boldsymbol{\mu}\cdot\mathbf{B}. \tag{4.50}$$

For an electron $\mathbf{S} = \frac{\boldsymbol{\sigma}}{2}$, and comparing with our expression for $\bar{u}\Gamma^\mu u(-ieA_\mu)$ (while accounting for the spinor normalization $2m_e$), we find

$$g_e = 2[F_1(0) + F_2(0)] = 2 + 2F_2(0). \tag{4.51}$$

4. The Standard Model at low energies

The deviation of the gyromagnetic ratio g_e from 2 is called the anomalous magnetic moment

$$a_e = \frac{g_e - 2}{2}. \tag{4.52}$$

It receives contributions from quantum corrections and from the operator O_{mag}, whose Wilson coefficient encapsulates the contribution from heavier states. Because it is sensitive to such corrections from heavier states, precision measurements of anomalous magnetic moments are used to search for physics beyond the SM.

Let us take a look at the different contributions to a_e. First, there are QED and hadronic contributions:

$\rightarrow \quad a_e = \frac{\alpha}{2\pi} = 10^{-3}$ [13] (Even the $\mathcal{O}(\alpha^5)$ corrections are known [14]!)

$\rightarrow \quad \frac{1}{45} \frac{m_e^2}{m_\mu^2} \left(\frac{\alpha}{\pi}\right)^2$

$\rightarrow \quad m_e^2 \triangle C_{\mathrm{mag}}^{\mathrm{hadronic}}$ (obtained from experimental data)
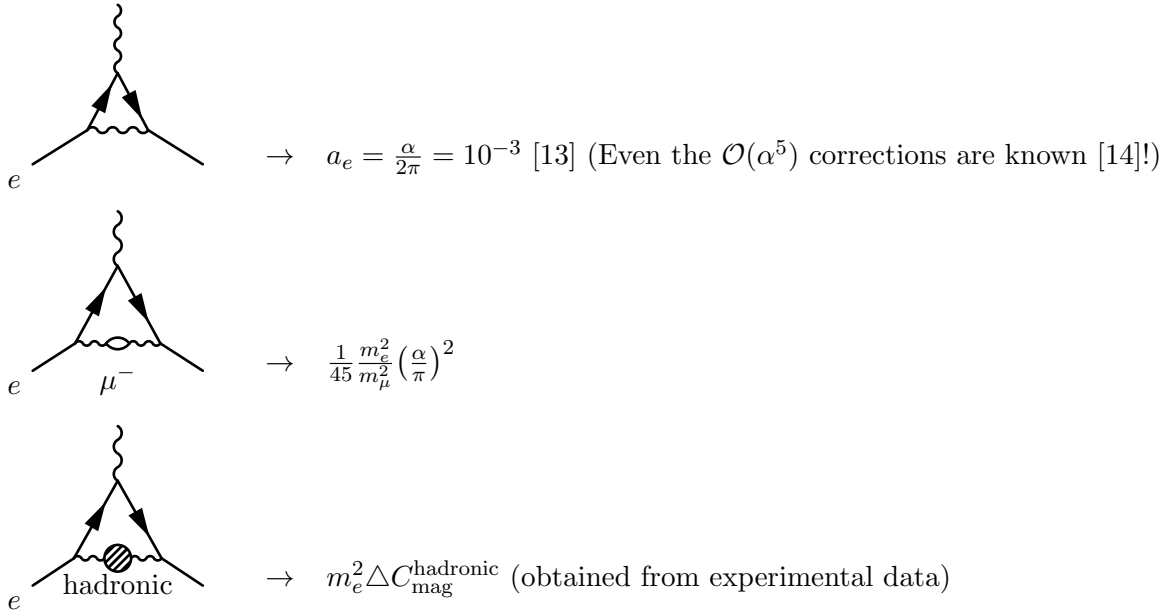
Next, there are electroweak corrections:

$\propto \frac{\alpha}{\pi} \frac{m_e^2}{M_W^2}$
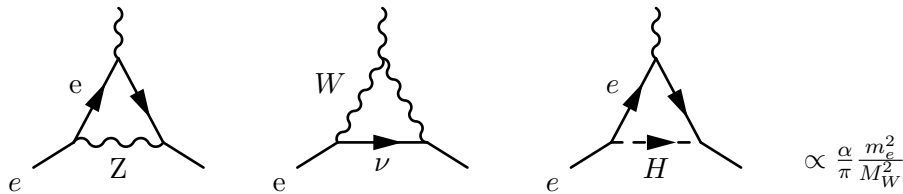
In addition to SM contributions, there could also be physics beyond the SM, e.g., supersymmetry (SUSY):

$\propto \frac{\alpha}{\pi} \frac{m_e^2}{\tilde{m}^2}$
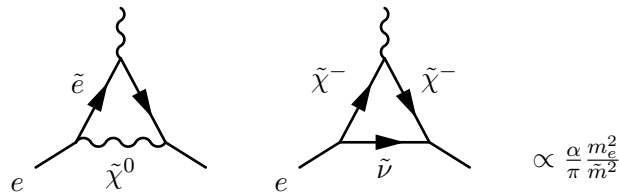
These contributions decouple with the heavy mass-scale $\tilde{m}$, but could lead to deviations from the SM prediction that can be detected in experiment.
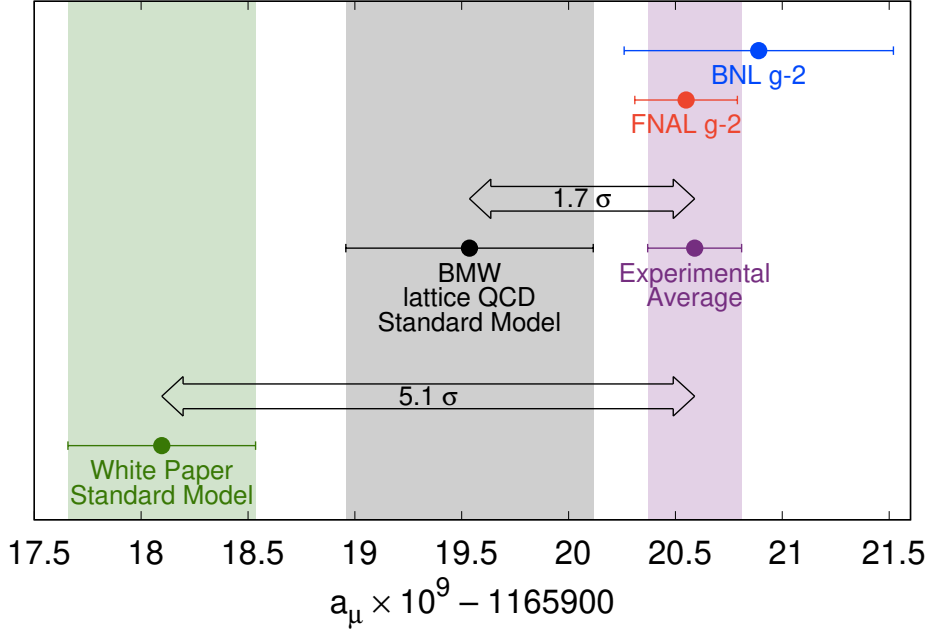
Figure 4.1.: Overview of the experimental measurement and theoretical prediction of the muon $g - 2$ as of 2024. The value

Since $(m_e/M_W)^2 \sim 4 \times 10^{-11}$, the weak interaction effects are very small for the electron $g - 2$. The effect of higher-mass particles are typically enhanced by $(\frac{m_\mu}{m_e})^2 \approx 4 \times 10^4$ in the *muon* $g - 2$. In 2021, the world average after the release of Run 1 data from the Fermilab E989 experiment [15] differed from the SM prediction in the White Paper [19] by 4.2σ, which was seen as a possible hint for contributions beyond the SM. In the meantime, as of 2024, the situation has become more complicated. With a new measurement [16], the deviation has increased to 5.1σ but at the same time, the picture on the theory side has become much more complicated. First of all, the first lattice determination of the hadronic vacuum polarization has appeared [17]. This is not as precise as the data-driven determination, but the resulting value of $g - 2$ is closer to the measured value. Furthermore, a new measurement of the $e^+e^- \rightarrow \pi^+\pi^-$ cross section was presented [18], which would move the theoretical prediction for $g - 2$ to a larger value, but is in conflict with earlier measurements. Until these discrepancies are sorted out, it is difficult to interpret these results.

4.2.2. Heavy flavors in QCD

The QCD Lagrangian with six quark flavors has the form

$$\mathcal{L}_{QCD} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} + \sum_{f=1}^{6} \bar{\psi}_f(i\not{D} - m_f)\psi_f \tag{4.53}$$

The quark field of each flavor has $N_c = 3$ components and the theory is invariant under gauge transformation in the group $SU(N_c)$ which rotates quarks of different colors into each other. We suppress the color indices of the quark fields and the covariant derivatives which

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are defined as

$$iD_\mu = i\partial_\mu \mathbf{1} + g_s A_\mu^a t^a \,. \tag{4.54}$$

Here $\mathbf{1}$ is unit matrix in color space and the matrices t^a, with $a = 1 \dots N_c^2 - 1$ are the generators of $SU(N_c)$. These generators fulfill the commutation relations

$$[t^a, t^b] = i f^{abc} t^c \tag{4.55}$$

where the structure constants f^{abc} encode the group structure. As in QED, the field strength tensor is defined through the commutator of covariant derivatives

$$[iD^a, iD^b] = i g_s G_{\mu\nu}^c t^c \,. \tag{4.56}$$

but in contrast to QED, it also contains a term quadratic in the field

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c \,, \tag{4.57}$$

which leads to three- and four-gluon interaction terms in the Lagrangian (4.53). It is useful to define

$$\alpha_s(\mu) = \frac{g_s^2(\mu)}{4\pi} \tag{4.58}$$

in analogy to the fine-structure constant in QED. In QED, one often uses a physical coupling α defined through the interaction of electrons at very low energies. Such a definition does not make sense in QCD. The coupling $\alpha_s(\mu)$ is instead defined in the $\overline{\text{MS}}$ scheme. As in the ϕ^4 example discussed in Section 3.5, the coupling fulfill an RG equation

$$\mu \frac{d\alpha_s(\mu)}{d\mu} = \beta(\alpha_s(\mu)) \tag{4.59}$$

which can be used to relate the values of the coupling constant at different scales μ.

The quark masses are very hierarchical and for many applications, one will need to integrate out the heavy flavors $m_t \simeq 173\,\text{GeV}$ and $m_b \simeq 5\,\text{GeV}$. The masses of these quarks are large enough that the matching can be performed perturbatively. For the charm quark with $m_c \simeq 1.3\,\text{GeV}$, $\alpha_s(m_c) \simeq 0.32$, this is still true, but the corrections will be quite significant. Let us discuss the effective theory obtained after integrating out the top quark. The $d = 4$ effective QCD Lagrangian has the same form as the original QCD Lagrangian, except that we now only include 5 flavors:

$$\mathcal{L}_{d=4} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} + \sum_{f=1}^{5} \bar{\psi}_f (i\slashed{D} - m_f)\psi_f. \tag{4.60}$$

If one neglects higher dimensional operators, the effective theory after integrating out quarks is simply QCD with one less flavor. Note that the QCD coupling runs differently for the $n_f = 5$ and $n_f = 6$ theories:

$$\mu \frac{d\alpha_s(\mu)}{d\mu} = \beta(\alpha_s(\mu)), \tag{4.61}$$

$$\beta(\alpha_s) = -2\alpha_s \left[\beta_0 \left(\frac{\alpha_s}{4\pi}\right) + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \dots\right], \tag{4.62}$$
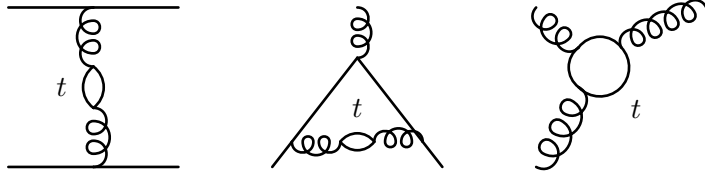
Figure 4.2.: Leading contributions to the Wilson coefficients of $O_{(Oi)}$, O_{mag} and O_3 when integrating out the top quark.

with $\beta_0 = (11C_A - 4n_f T_F)/3$ and $\beta_1 = 2\big[17C_A^2 - n_f T_F(10C_A + 6C_F)\big]/3$. If one goes to higher loop orders, one also has to perform a nontrivial matching computation to obtain the coupling constant α_s in the low energy theory without the heavy flavor from the one at higher energies which includes it, as will be discussed below.

However, let us first discuss higher-dimensional operators in the effective theory. All the operators found in QED are also allowed in QCD, so if one includes higher dimensional operators then at $d = 6$, we find the same four-fermion operators as in the QED case, but with two different color structures, a first one, where the quark bilinears are in a color singlet (S) configuration

$$O_{(Si)}^{ff'} = \bar{\psi}_f \gamma^{[\mu_1} \ldots \gamma^{\mu_i]} \psi_f \bar{\psi}_{f'} \gamma_{[\mu_1} \ldots \gamma_{\mu_i]} \psi_{f'}, \tag{4.63}$$

and a second one, where they form a color octet (S)

$$O_{(Oi)}^{ff'} = \bar{\psi}_f t^a \gamma^{[\mu_1} \ldots \gamma^{\mu_i]} \psi_f \bar{\psi}_{f'} t^a \gamma_{[\mu_1} \ldots \gamma_{\mu_i]} \psi_{f'}. \tag{4.64}$$

This second structure is the same that arises if one starts with $O_{(Si)}^{ff'}$ and exchanges a gluon between the two quark bilinears. Since the Wilson coefficients of the operators with i even will be suppressed by quark masses, it is again sufficient to consider $i = 1$ and $i = 3$ at dimension 6.

Since QCD does not distinguish the flavors,[3] the Wilson coefficients of $O_{(5i)}^{ff'}$, $O_{(0i)}^{ff'}$ are independent of the flavor indices, i.e., we only need the two operators

$$O_{(Si)} = \sum_{ff'} O_{(Si)}^{ff'}, \qquad O_{(Oi)} = \sum_{ff'} O_{(Oi)}^{ff'}. \tag{4.65}$$

As in QED, we also get a magnetic operator (which is called chromo-magnetic, since it involves the gluon field)

$$O_{\text{mag}} = \sum_f m_f \bar{\psi}_f \sigma^{\mu\nu} G_{\mu\nu}^a t^a \psi_f, \tag{4.66}$$

but there is one additional operator not present in QED, namely

$$O_3 = f^{abc} G_{\mu\nu}^a G_b^{\nu\sigma} G_\sigma^{c,\mu}, \tag{4.67}$$

which arises from the non-Abelian nature of QCD. The leading contributions to the Wilson coefficients of these operators originate from the diagrams shown in Figure 4.2.

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[3]Except for the quark masses, which can be set to zero for the matching computation
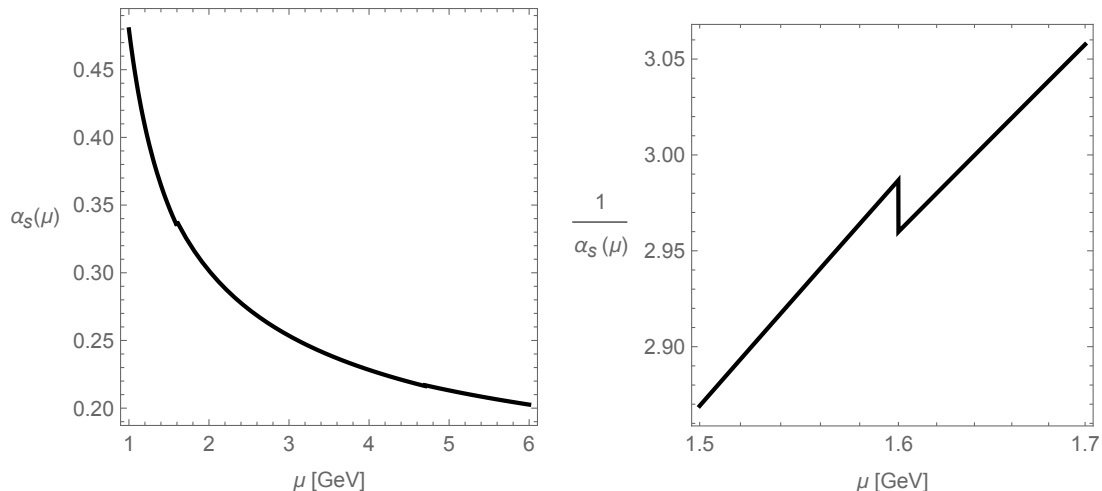
Figure 4.3.: Running of the strong coupling constant $\alpha_s(\mu)$ obtained using the RunDec package [20, 21]. The right plot shows the transition from QCD with $n_f = 4$ quark flavors to QCD with $n_f = 3$ at the matching scale $\mu_m = m_c = 1.6\,\text{GeV}$. The discontinuity arises from (4.71).

Let us now discuss the matching for $\mathcal{L}_{d=4}$, which contains as Wilson coefficient the coupling constant g_s and m_f, and concentrate on $\alpha_s(\mu) = \frac{g_s^2(\mu)}{4\pi}$. We denote the coupling by $\alpha_s^{(n_f)}(\mu)$ to distinguish α_s in the theory with $n_f = 6$ from $\alpha_s^{(5)}(\mu)$ in the theory where the top quark is integrated out. Then we proceed as in our scalar toy model. At a scale $\mu_m \approx m_t$ one derives a matching relation

$$\alpha_s^{(5)}(\mu_m) = \alpha_s^{(6)}(\mu_m)\,\xi_A^{-1}\big[\alpha_s^{(6)}(\mu_m)\big]\,, \tag{4.68}$$

which gives the coupling constant in the low energy theory with $n_f = 5$ as an expansion of the one in six-flavor QCD. The simplest way to obtain ξ_A is to compute the gluon propagator in both theories. The pole term in the propagator has the form

$$G_{\mu\nu}(q^2) = \frac{iZ_A^{\text{os}}}{q^2}(-g_{\mu\nu} + \dots) \tag{4.69}$$

where Z_A^{os} is the gluon on-shell wave function renormalization constant. Rescaling the coupling by ξ_A is the same as rescaling the gluon field. One can then show that

$$\xi_A^{(0)} = \frac{Z_A^{\text{os},(6)}}{Z_A^{\text{os},(5)}} \quad \text{with} \quad Z_A^{\text{os}} = \frac{1}{1 - \Pi(0)}, \tag{4.70}$$

where $\Pi(0)$ is the polarization function evaluated at virtuality $q^2 = 0$. If one chooses $\mu_m = m_t(\mu_m)$ the expression for ξ_A is especially simple

$$\xi_A(m_t) = 1 + \left(\frac{13}{3}C_F - \frac{32}{9}C_A\right)T_F\left(\frac{\alpha(m_t)}{4\pi}\right)^2, \tag{4.71}$$

where $C_F = (N_c^2 - 1)/(2N_c) = 4/3$, $C_A = N_c = 3$ are the Casimir invariants of the fundamental and adjoint representations, respectively, and $T_F = 1/2$ is the normalization of the

generators. The running coupling at low values of μ is shown in the left plot in Figure 4.3. The right plot in the same figure shows the transition from QCD with a dynamical charm quark to the effective theory obtained by integrating out at $\mu_m = m_c = 1.6\,\text{GeV}$. The jump in the coupling is due to the matching correction in (4.71). Plotting this jump is slightly misleading since the coupling constants above and below $\mu_m = m_c$ belong to different theories. The matching correction (4.71) is obtained by ensuring the two theories give the same result at $\mu = m_c$; while the number of flavors n_f and α_s are discontinuous at this point, the observables in the two theories agree and are of course continuous under variation of μ.

4.3. Effective weak Hamiltonian

Let us now discuss the weak interactions at low energies. In the SM, the weak and electromagnetic interactions are described by a $SU(2)_L \times U(1)_Y$ gauge theory, which is broken down to $U(1)_{\text{em}}$ by the Higgs mechanism, which gives masses to the $W^\pm$ and Z^0 bosons. A detailed introduction to the SM is beyond the scope of this lecture. The only information needed for our discussion is the charged-current coupling of $W^\pm$ bosons to fermions. It has the form

$$\mathcal{L}_{\text{cc}} = -\frac{g_2}{2\sqrt{2}}(J_\mu^+ W^{+\mu} + J_\mu^- W^{-\mu}), \tag{4.72}$$

where

$$J_\mu^+ = (J_\mu^-)^\dagger = (\bar{u}d')_{V-A} + (\bar{c}s')_{V-A} + (\bar{t}b')_{V-A} + (\bar{\nu}_e e)_{V-A} + (\bar{\nu}_\mu \mu)_{V-A} + (\bar{\nu}_\tau \tau)_{V-A} \tag{4.73}$$

and $(\bar{u}d')_{V-A} = \bar{u}\gamma_\mu(1 - \gamma_5)d'$, etc. The quark fields d', s', and b' do not correspond to mass eigenstates, i.e., the quadratic part of $\mathcal{L}_{\text{SM}}$ is not diagonal. The Cabibbo–Kobayashi–Maskawa (CKM) matrix connects (d', s', b') to the mass eigenstates (d, s, b)

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = V_{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{4.74}$$

The matrix is unitary, so $V_{\text{CKM}} V_{\text{CKM}}^\dagger = \mathbb{1}$. It can be further simplified by phase redefinitions of the fermion fields in $\mathcal{L}_{\text{cc}}$ and has $3 \times 3 - (2 \times 3 - 1) = 4$ physical parameters. Its structure is reflected by the Wolfenstein parameterization, which was designed to show the hierarchy of the different matrix elements:

$$\hat{V}_{\text{CKM}} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \tag{4.75}$$

with

$$\lambda \simeq 0.225, \quad \rho \simeq 0.139, \quad A \simeq 0.81, \quad \eta \simeq 0.342.$$

This form is only approximately unitary, up to higher orders in λ. The parameters of the matrix correspond to three rotations and one complex phase, which leads to CP violation. Similarly, the neutrinos ν_e, ν_μ, and ν_τ are not mass eigenstates. The corresponding mixing matrix is called PMNS matrix (Pontecorvo–Maki–Nakagawa–Sakala), but does not play a role in what follows.

4. The Standard Model at low energies
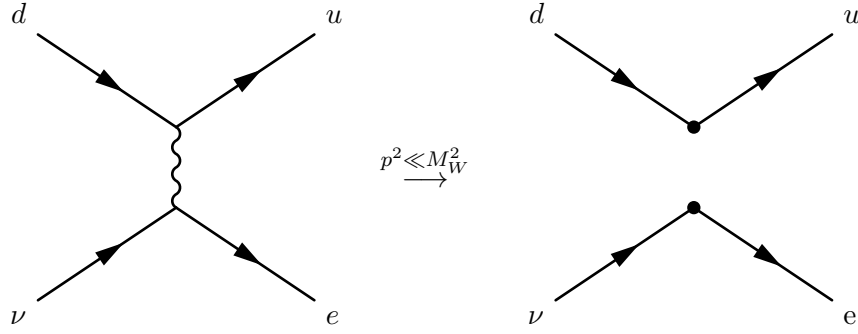
Figure 4.4.: Expanding the W propagator at small external momentum yields four-quark operators.

Let us first work at tree level and neglect QCD effects. Then the effective weak Lagrangian can be obtained by integrating out the $W^{\pm}$ and Z fields. The resulting effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = -\frac{g_2^2}{8M_W^2}\left[J_\mu^- J^{+\mu} + \frac{1}{M_W^2}J_\mu^-(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)J_\nu^+ + \dots\right] \qquad (4.76)$$

where $\frac{g_2^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}$ defines the Fermi constant $G_F = 1.166 \times 10^{-5}\,\text{GeV}^{-2}$. Diagrammatically, this arises from expanding the W-propagator in the weak interaction diagrams for small external momentum as

$$\frac{-i}{p^2 - M_W^2}\left[g^{\mu\nu} - \frac{p^\mu p^\nu}{M_W^2}\right] = \frac{i}{M_W^2}\left[g^{\mu\nu} - \frac{1}{M_W^2}(p^\mu p^\nu - p^2 g^{\mu\nu}) + \mathcal{O}\left(\frac{1}{M_W^4}\right)\right], \qquad (4.77)$$

so that already the leading terms in $\mathcal{L}_{\text{eff}}$ are irrelevant operators of $d = 6$ (recall that the fermion field has $d = 3/2$). Indeed the coefficient of the four-fermion operators $G_F \sim \frac{1}{M_W^2}$ shows the expected behavior. The fact that these are not marginal or relevant operators explains the apparent weakness of the interaction at low energies. At high energies, on the other hand, the weak-interaction effects are as strong as electromagnetic interactions. Because of the $\frac{1}{M_W^2}$ suppression, the leading-order $d = 6$ terms are good enough for most applications. Since it changes lepton and quark flavors, $\mathcal{L}_{\text{eff}}$ governs all decays of heavy leptons and hadrons, such as

$$\begin{aligned}
\mu^- &\to e^- + \bar{\nu}_e + \nu_\mu, \\
\pi^- &\to \mu^- + \bar{\nu}_\mu, \\
n &\to p + e^- + \bar{\nu}_e.
\end{aligned} \qquad (4.78)$$

The Particle Data Group (PDG) lists hundreds of pages of various hadron decays [23]. In the SM all such decays that proceed via the weak interactions are governed by G_F and the four parameters in the CKM matrix. If one manages to evaluate the strong-interaction effects in such decays, they offer many opportunities to search for physics beyond the SM. An important step is to include QCD corrections to the effective Lagrangian. To do so, one has to

1. include a complete set of operators, not only those present at tree level,

2. perform a matching computation to obtain the Wilson coefficients,
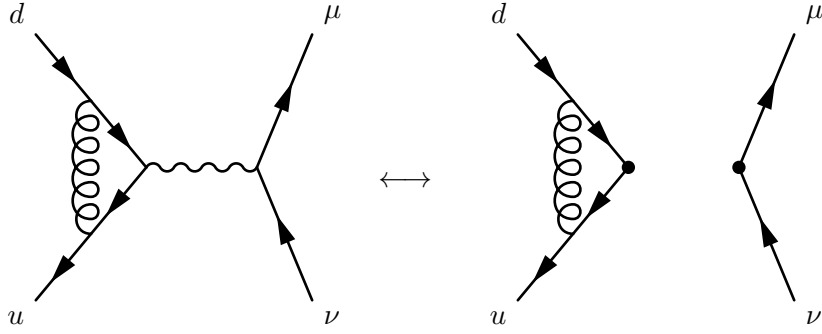
Figure 4.5.: One-loop QCD corrections in the SM and in the weak effective theory. The figure illustrates that the corrections are identical in both theories.

3. solve the RG equation for the coefficients to resum large logs.

We will now discuss two examples that illustrate how the construction works in the general case.

4.3.1. Leptonic decays

Let us consider the operator relevant for $\pi^- \to \mu^- + \bar{\nu}_\mu$, which is based on the quark-level transition $\bar{u}d \to \bar{\nu}_\mu \mu^-$. The tree-level $\mathcal{L}_{\text{eff}}$ is

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{ud} (\bar{u}d)_{V-A} (\bar{\mu}\nu_\mu)_{V-A}. \tag{4.79}$$

It turns out that this is the only operator with this flavor structure: the weak interactions only couple to left-handed fields $\psi_L = \frac{1}{2}(1 - \gamma_5)\psi$ and the QCD interactions conserve helicity for vanishing quark masses $\mathcal{L}_{\text{QCD}} = \bar{\psi}_L i \slashed{D} \psi_L + \bar{\psi}_R i \slashed{D} \psi_R$. Chirality-violating operators are suppressed by powers of the quark masses. The only possible Dirac bilinears are $\bar{\psi}_L \gamma^\mu \psi_L = -\bar{\psi}_L \gamma^\mu \gamma^5 \psi_L$ and $\bar{\psi}_L \psi_L = \bar{\psi}_L \sigma^{\mu\nu} \psi_L = 0$. This leaves only the operators

$$\bar{u}_L \gamma^\mu d_L \, \bar{\mu}_L \gamma_\mu \nu_L = \bar{u}_L \gamma^\mu \nu_L \, \bar{\mu}_L \gamma_\mu d_L. \tag{4.80}$$

The fact that the two operators are equal is an example of Fierz identities, which follow from a rearrangement of the Dirac structures.

Not only is there just a single operator, but also the matching is trivial since all QCD corrections are the same in the full theory and the effective theory, see Figure 4.5. Accordingly, there are no QCD corrections to $\mathcal{L}_{\text{eff}}$. QCD effects only arise in the matrix element

$$\langle 0| \, \bar{u}(0)\gamma^\mu(1-\gamma_5)d(0) \, |\pi^-(p)\rangle = -\langle 0| \, \bar{u}\gamma^\mu\gamma_5 d \, |\pi^-(p)\rangle$$
$$= -i f_\pi p^\mu = -i\sqrt{2} F_\pi p^\mu. \tag{4.81}$$

Since the matrix element is a Lorentz vector and only involves a single vector p^μ it must be proportional to it, and since $p^2 = M_\pi^2$ is constant it cannot have any kinematic dependence. The quantity F_π is called the pion decay constant. By measuring the pion decay rate

$$\Gamma(\pi \to \mu\nu) = \frac{G_F^2}{4\pi} F_\pi^2 m_\mu^2 M_\pi \left(1 - \frac{m_\mu^2}{M_\pi^2}\right) |V_{ud}|^2 \tag{4.82}$$

one can then determine the combination $|V_{ud}|F_\pi$. Similarly, one obtains $|V_{cd}|$ from $D^- \to \mu^- \bar{\nu}$, $|V_{ub}|$ from $B^- \to \tau^- \bar{\nu}$, and $|V_{us}|$ from $K^- \to \mu^- \bar{\nu}$, in combination with the respective decay constants. To turn the measurement of the decays into determinations of the CKM matrix elements, one needs independent computations of the decays constants which can be obtained using lattice QCD, see [22] for a compilation of such results.

4.3.2. Hadronic decays

Let us consider next the decay $\bar{B}^{(0)} \to D_s^- \pi^+$, based on the $b \to u\bar{c}s$ quark-level transition. In this case, there are two operators that differ by their color structure

$$
\begin{aligned}
O_1 &= \bar{s}_L^i \gamma_\mu c_L^i \bar{u}_L^j \gamma^\mu b_L^j, \\
O_2 &= \bar{s}_L^i \gamma_\mu c_L^j \bar{u}_L^j \gamma^\mu b_L^i,
\end{aligned}
\tag{4.83}
$$

where the color indices i and j are summed over. Note that

$$
\bar{s}_L \gamma_\mu t^a c_L \bar{u}_L \gamma^\mu t^a b_L = \frac{1}{2} O_2 - \frac{1}{2N_c} O_1,
\tag{4.84}
$$

which follows from the identity

$$
t_{ij}^a t_{kl}^a = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right)
\tag{4.85}
$$

for the $SU(3)_c$ generators.

The low-energy effective Lagrangian for the $b \to u\bar{c}s$ transition thus takes the form

$$
\mathcal{L}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{cs}^* V_{ub} \Big[C_1(\mu) O_1 + C_2(\mu) O_2 \Big],
\tag{4.86}
$$

where the coefficients encode the high-energy QCD corrections. Depending on the quark transitions under consideration, the weak effective Lagrangian will contain more operators. Our Lagrangian is particularly simple because it does not get contributions from neutral current interactions since all four quark flavors in the operator are different. A review of the relevant Lagrangians can be found in [27] and a pedagogical introduction in [26]. The Lagrangian (4.86) is the simplest example to discuss the matching and renormalization in a case where we have several operators with the same quantum numbers. This is interesting because the different operators mix under renormalization, which is also reflected in the associated RG equation. This provides us with a nontrivial example of RG improved perturbation theory discussed in Section 3.5.

Let us start with the matching of the full onto the effective theory. At tree level one has $C_1 = 1$ and $C_2 = 0$, which follows from the same argument as in the leptonic example discussed before. To obtain the one-loop coefficients, however, one has to perform a matching computation, i.e., one has to compute the $b \to u\bar{c}s$ transition in both the full and the effective theory. In Figure 4.6 we show the associated one-loop diagrams in both the full and effective theory.

The difference between the full and the effective-theory results is absorbed into C_1 and C_2. Since C_1, C_2 are independent of m_q, we can set all quark masses to zero. Furthermore, also any values for the external momenta will work, with the simplest choice $p_i = 0$. In this case
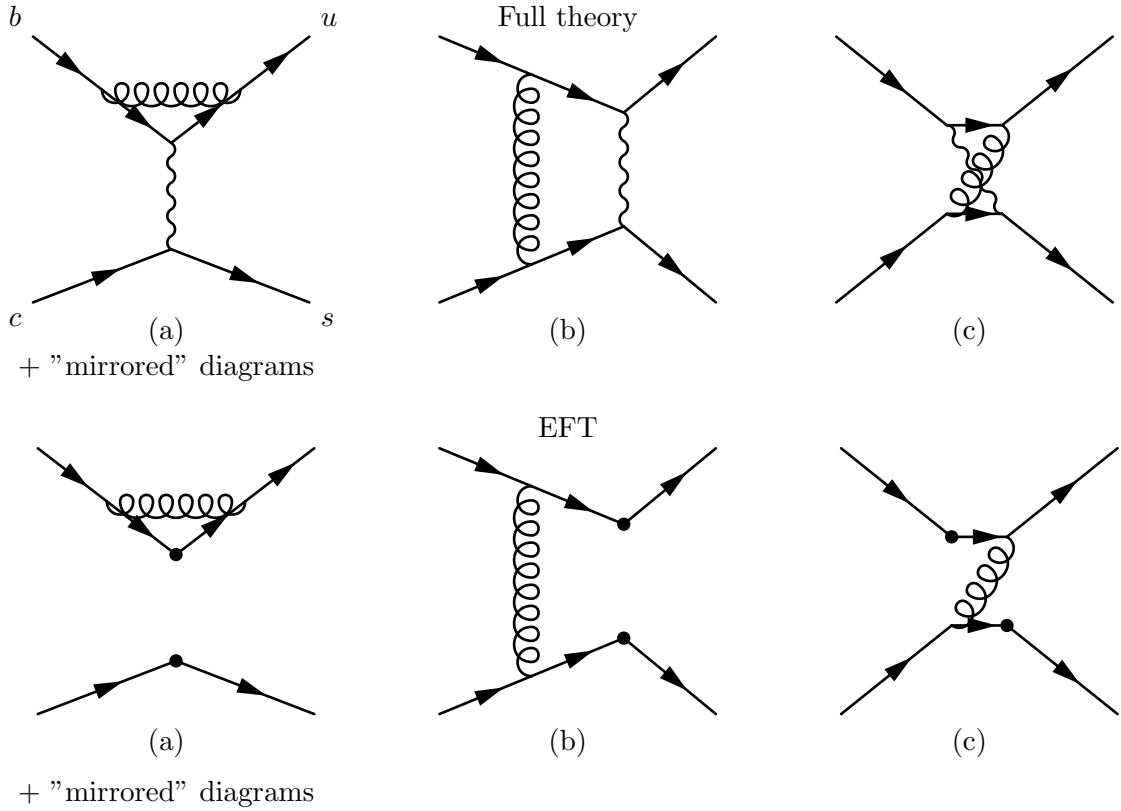
Figure 4.6.: One-loop diagrams in the matching of the $b \to u\bar{c}s$ transition onto (4.86).
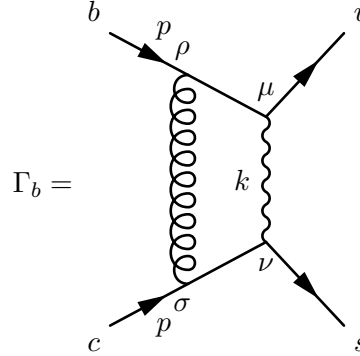
diagram (a) in the full theory and all diagrams in the effective theory vanish, because they are scaleless

$$\int d^d k \frac{1}{k^2} \frac{1}{\not{k}} \Gamma \frac{1}{\not{k}} \Gamma = 0 = \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}}, \tag{4.87}$$

which amounts to a cancellation of IR and UV divergences in dimensional regularization. Since the IR divergences are a low-energy property of the theory, they are present in both the full and the effective theory, and thus cancel in the matching. We will keep the p_i non-zero, but use the same value for all legs. This makes the all matrix elements infrared finite, which has the advantage that we can also extract the renormalization factors from our computations.

Let us start with diagram (b) in the full theory

4. The Standard Model at low energies

$$\Gamma_b =$$

$$= i^2 \frac{G_F M_W^2}{\sqrt{2}} V_{cs}^* V_{ub} \int \frac{d^d k}{(2\pi)^d} \frac{-i}{k^2} \frac{-i}{k^2 - M_W^2} \frac{i}{(p+k)^2} \frac{i}{(p-k)^2}$$
$$\times \bar{u}\gamma^\mu (1-\gamma_5)(\slashed{p}+\slashed{k}) i g_s t^a \gamma^\rho b \times \bar{s}\gamma^\nu (1-\gamma_5)(\slashed{p}-\slashed{k}) i g_s t^a \gamma^\sigma c \tag{4.88}$$
$$\times g_{\rho\sigma} \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2} \right].$$

Some remarks are in order:

1. We compute amputated Green's function but write down external spinors $\bar{u}\Gamma_1 b\, \bar{s}\Gamma_2 c$ to remind ourselves which color and spin index goes where.

2. We are using Feynman gauge for QCD and unitary gauge for the W-propagator. The $k^\mu k^\nu / M_W^2$ term does not contribute to the sum of the diagrams, so we will omit it.

3. The color structure is $t^a \otimes t^a$. To rewrite this in the form of O_1 and O_2, we use the identity (4.85).

4. To simplify the Dirac structure one needs identities such as $[\Gamma = \gamma^\alpha (1 - \gamma^5)]$

$$\Gamma \gamma_\beta \gamma_\mu \otimes \Gamma \gamma^\beta \gamma^\mu = 16 \Gamma \otimes \Gamma. \tag{4.89}$$

The coefficient can be derived by taking traces $\mathrm{Tr}[A\Gamma\gamma_\beta\gamma_\mu B\Gamma\gamma^\beta\gamma^\mu]$, $\mathrm{Tr}[A\Gamma B\Gamma]$ for some Dirac matrices A and B, e.g., $A = B = \gamma^5$.

5. Without the $k^\mu k^\nu / M_W$ term, the diagram is finite, so we only need its value for $d = 4$.

The diagram (b) in the effective theory, on the other hand, is divergent, as it behaves like

$$\int d^d k \frac{1}{k^2} \frac{1}{\slashed{p} - k} \frac{1}{\slashed{p} + k} \sim \int \frac{d^d k}{k^4} \sim \frac{1}{\epsilon} + \log \frac{\mu^2}{p^2}. \tag{4.90}$$

This leads to some technical issues:

1. The Dirac basis can be thought of as all totally antisymmetric products of γ matrices. In d dimensions, one can also write down antisymmetric products of more than four γ matrices, which are not necessary in four dimensions. Operators with such Dirac structures are called *evanescent*. One can use a renormalization scheme in which their physical matrix elements vanish, but they need to be included in $\mathcal{L}_{\mathrm{eff}}$ for consistency, see [9] for details.

2. γ^5 is special to $d = 4$. The rule $\{\gamma^\mu, \gamma^5\} = 0$ leads to inconsistencies in d dimensions. A prescription to eliminate γ^5 is the 't Hooft/Veltman scheme [24], which can be thought of as replacing $\gamma_5 = -\frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, rewriting two epsilon tensors in terms of metric tensors using (4.20), and only dealing with a potentially remaining epsilon tensor *after* renormalization. Alternatively, one can use the so-called NDR scheme (naive dimensional regularization), i.e., use $0 = \{\gamma^\mu, \gamma^5\}$ and hope that no inconsistency arise, or the DRED scheme (dimensional reduction), i.e., treat γ^μ and the gauge fields as four dimensional. A review of the different schemes can be found in [25].

Let us now discuss the results. The bare Green's function in the SM and in the EFT are ($d = 4 - 2\epsilon$):

$$\Gamma_{\text{full}} = \frac{G_F V_{cs}^* V_{ub}}{\sqrt{2}}\left\{\left(1 + 2C_F\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)\right)\langle O_1\rangle_{\text{tree}} + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{-p^2}\langle O_1\rangle_{\text{tree}}\right.$$
$$\left. - 3\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{-p^2}\langle O_2\rangle_{\text{tree}}\right\} + \mathcal{O}\left(\frac{p^2}{M_W^2}\right), \tag{4.91}$$

$$\Gamma_{\text{eff}} = \frac{G_F V_{cs}^* V_{ub}}{\sqrt{2}}\left\{C_1^{\text{bare}}\left[\left(1 + 2C_F\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)\right)\langle O_1\rangle_{\text{tree}}\right.\right.$$
$$\left. + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)\langle O_1\rangle_{\text{tree}} - 3\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)\langle O_2\rangle_{\text{tree}}\right]$$
$$+ C_2^{\text{bare}}\left[\left(1 + 2C_F\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)\right)\langle O_2\rangle_{\text{tree}} + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)\langle O_2\rangle_{\text{tree}}\right.$$
$$\left.\left. - 3\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)\langle O_1\rangle_{\text{tree}}\right]\right\} + \mathcal{O}\left(\frac{p^2}{M_W^2}\right). \tag{4.92}$$

In each case, the first term corresponds to diagram (a), the remainder to (b)+(c) (and their mirrored counterparts). The color factors are $N_c = 3$ and $C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$, $\alpha_s = \frac{g_s^2}{4\pi}$. The Wilson coefficients have an expansion $C_i = C_i^{(0)} + \frac{\alpha_s}{4\pi}C_i^{(1)} + \ldots$, as reflected by the α_s corrections in Eq. (4.91). The bare amputated Green's functions have ultraviolet divergences, corresponding to the $1/\epsilon$ poles. In the full theory, these divergences are removed by the wave-function renormalization $\psi^{(0)} = Z_q^{1/2}\psi$, with the one-loop wave function renormalization $Z_q = 1 - \frac{\alpha_s}{4\pi\epsilon}C_F$ in $\overline{\text{MS}}$ and for Feynman gauge. In the effective theory, there are additional divergences from (b) and (c), which are not removed by wave-function renormalization of the Wilson coefficients in $\mathcal{L}_{\text{eff}}$. Omitting the overall factor $-\frac{G_F}{\sqrt{2}}V_{cs}^* V_{ub}$, one has

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}}V_{cs}^* V_{ub} C_i^{\text{bare}} O_i(q^{(0)}) = -\frac{G_F}{\sqrt{2}}V_{cs}^* V_{ud} Z_q^2 C_i Z_{ij} O_j(q). \tag{4.93}$$

The additional renormalization constants form a *matrix Z_{ij}*. Since the two operators have the same quantum numbers, they cannot be renormalized separately. The fact that the matrix Z_{ij} is non-diagonal is referred to as operator mixing. Expanding $Z_{ij} = \delta_{ij} + \frac{\alpha_s}{4\pi\epsilon}Z_{ij}^{(1)}$, one finds that

$$\hat{Z} = \mathbb{1} + \frac{\alpha_s}{4\pi\epsilon}\begin{pmatrix} -3/N_c & 3 \\ 3 & -3/N_c \end{pmatrix} \tag{4.94}$$

removes the remaining divergence in Γ_{eff}. We see that we indeed encounter operators mixing so that the two operators cannot be separately renormalized.

From the condition $\Gamma_{\text{full}} - \Gamma_{\text{eff}} = 0$, one can then finally read off C_1 and C_2:

$$C_1(\mu) = 1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{\mu^2}, \tag{4.95}$$

$$C_2(\mu) = 0 - 3\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{\mu^2}. \tag{4.96}$$

The crucial point here is that C_1 and C_2 only depend on M_W and μ, but not on the low-energy scale p^2. This has to be the case, as the low-energy physics must drop out in the matching. The last step in the construction of $\mathcal{L}_{\text{eff}}$ is to solve the RG equations for C_1 and C_2 to avoid having large logarithms when evaluating $C_1(\mu)$ at low values of μ. The RG equations follow from the fact that physical quantities are μ independent. Equivalently, we can use the fact that bare quantities are μ independent:

$$\mu\frac{d}{d\mu}C_j^{\text{bare}}(\epsilon, \alpha^{\text{bare}}) = 0 = \mu\frac{d}{d\mu}C_i(\mu)Z_{ij}(\alpha_s(\mu), \epsilon) \tag{4.97}$$

$$= \left(\mu\frac{d}{d\mu}C_i(\mu)\right)Z_{ij} + C_i(\mu)\left(\mu\frac{d}{d\mu}Z_{ij}\right), \tag{4.98}$$

so that

$$\mu\frac{d}{d\mu}C_j(\mu) - C_i(\mu)\,\gamma_{ij}(\alpha) = 0, \tag{4.99}$$

where

$$\gamma_{ij} = -(\mu\frac{d}{d\mu}Z_{ik})Z_{kj}^{-1} \tag{4.100}$$

is the anomalous dimension matrix associated with the two operators. In vector notation (4.99) can be written as

$$\left(\mu\frac{d}{d\mu} - \hat{\gamma}^T\right)\vec{C}(\mu) = 0. \tag{4.101}$$

In the $\overline{\text{MS}}$ scheme, the Z-matrix is a sum of pole terms

$$\hat{Z} = \mathbb{1} + \sum_{k=1}^{\infty}\frac{1}{\epsilon^k}\hat{Z}_{[k]}(\alpha_s) \tag{4.102}$$

and there is a simple relation (also known as the "magic relation") to obtain the anomalous dimension directly from the $1/\epsilon$ pole of the Z-matrix. It reads

$$\hat{\gamma} = 2\alpha_s\frac{\partial\hat{Z}_{[1]}}{\partial\alpha_s} = \frac{\alpha_s}{4\pi}\begin{pmatrix} -6/N_c & 6 \\ 6 & -6/N_c \end{pmatrix}. \tag{4.103}$$

To solve the RG equation, it is simplest to use a basis in which $\hat{\gamma}$ is diagonal, in our case the corresponding combinations are $C_\pm = C_1 \pm C_2$, with

$$\mu\frac{d}{d\mu}C_\pm(\mu) = \frac{\alpha_s}{4\pi}6\left(\pm 1 - \frac{1}{N_c}\right)C_\pm(\mu) \tag{4.104}$$

$$\equiv \frac{\alpha_s}{4\pi}\gamma_\pm C_\pm(\mu). \tag{4.105}$$
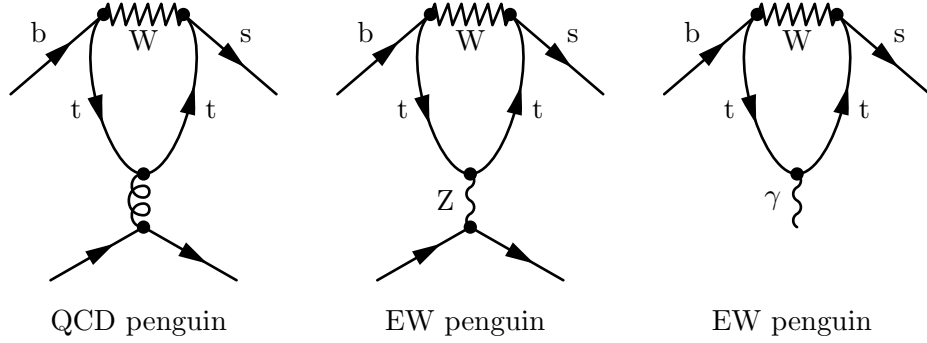
Figure 4.7.: Examples of penguin diagrams. See [29] for details on the origin of the name.

Converting the derivatives using the QCD β function

$$\mu \frac{d\alpha_s}{d\mu} = -2\alpha_s \left[\frac{\alpha_s}{4\pi} \beta_0 + \left(\frac{\alpha_s}{4\pi} \right)^2 \beta_1 + \cdots \right], \tag{4.106}$$

we can solve these RG equations by a separation of variables

$$\frac{dC_\pm}{C_\pm} = \frac{d\mu}{\mu} \frac{\alpha_s}{4\pi} \gamma_\pm = \frac{d\alpha_s}{\beta(\alpha_s)} \frac{\alpha_s}{4\pi} \gamma_\pm = -\frac{d\alpha_s}{\alpha_s} \frac{\gamma_\pm}{2\beta_0} + \cdots, \tag{4.107}$$

with the result

$$\log \frac{C_\pm(\mu)}{C_\pm(M_W)} = -\frac{\gamma_\pm}{2\beta_0} \log \frac{\alpha_s(\mu)}{\alpha_s(M_W)} \tag{4.108}$$

or

$$C_\pm(\mu) = C_\pm(M_W) \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{-\frac{\gamma_\pm}{2\beta_0}}. \tag{4.109}$$

Using $C_\pm(M_W) = 1 + \mathcal{O}(\alpha_s)$ and rotating back to the original basis we find

$$C_1(\mu) = \frac{1}{2} \left[\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{-\frac{\gamma_+}{2\beta_0}} + \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{-\frac{\gamma_-}{2\beta_0}} \right], \tag{4.110}$$

$$C_2(\mu) = \frac{1}{2} \left[\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{-\frac{\gamma_+}{2\beta_0}} - \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{-\frac{\gamma_-}{2\beta_0}} \right], \tag{4.111}$$

where

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f, \qquad \gamma_\pm = 6 \left(\pm 1 - \frac{1}{N_c} \right) = \mp 6 - 2. \tag{4.112}$$

Numerically, one finds for $\mu = m_b$

$$C_1(\mu) = 1.10, \quad C_2(\mu) = -0.24, \tag{4.113}$$

so that at the low scale indeed a non-vanishing result for C_2 has been induced, while the value of C_1 changes by a similar amount. This completes our discussion the QCD effects in $\mathcal{L}_{\text{eff}}$ relevant for $b \to u\bar{c}s$, which mediates $\bar{B} \to D_s^- \pi^+$.

As mentioned above, the structure of $\mathcal{L}_{\text{eff}}$ becomes more complicated for flavor-changing-neutral current (FCNC) processes such as

$$b \to s\gamma, \qquad b \to sg, \qquad b \to s\bar{q}q, \qquad b \to sl^+l^-. \tag{4.114}$$

For such decays so-called penguin diagrams contribute, examples of which are shown in Figure 4.7. The effective theory operators associated with these are obtained by contracting all internal lines to points.

Such processes are interesting because they can violate CP symmetry and are sensitive to new physics because they only arise at loop-level in the SM. The effective Lagrangian for such processes contains ~ 12 operators, which all mix under renormalization, i.e.,

$$\left[\mu\frac{d}{d\mu} - \hat{\gamma}^T\right]\vec{C} = 0. \tag{4.115}$$

Although there is a lot of interesting phenomenology associated with such decays, we will not discuss them further, but let us note that the corresponding anomalous-dimension matrix has been calculated at $\mathcal{O}(\alpha_s^3)$, which involved the computation of hundreds of thousands of three- and four-loop diagrams.

4.4. Chiral perturbation theory

Let us finally turn to the strong interaction at low energies. Instead of quarks and gluons, the observed particles are hadrons, i.e., mesons such as π, K, η, η', ρ, ... and baryons p, n, Δ, Σ, ... The effective Lagrangian is then a function of hadron fields. As in all our previous applications, one starts by writing down the most general $\mathcal{L}_{\text{eff}}$ compatible with the symmetries of the underlying theory, i.e., QCD. In contrast to previous examples, however, we will be unable to perform matching computations due to our limited ability to perform QCD computations at low energy (using lattice-QCD simulations, it is becoming possible to some extent). At first sight, it looks like an effective theory of hadrons and it will be not very predictive since the Wilson coefficients are not known. However, it turns out that chiral symmetry severely constrains the interactions of the light hadrons, and the EFT approach is very useful to derive the consequences of this approximate symmetry.

4.4.1. Chiral Symmetry

Since we will work at very low energies, we can integrate out the heavy-quark flavors and use

$$\mathcal{L}_{\text{QCD}}^{\text{eff}} = -\frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu} + \sum_{f=u,d,s} \bar{q}_f(i\slashed{D} - m_f)q_f + \mathcal{O}\left(\frac{1}{m_{c,b,t}^2}\right). \tag{4.116}$$

The theory simplifies further in the *chiral limit* $m_q \to 0$. Since only the mass term distinguishes different flavors, a new flavor symmetry arises. In fact, the symmetry group is even larger. To see this we split the fermion fields into their left- and right-handed parts $q_{L,R} = P_{L,R}\,q$, with $P_L = \frac{1}{2}(1 - \gamma_5)$, $P_R = \frac{1}{2}(1 + \gamma_5)$. The Lagrangian then reads

$$\mathcal{L}_{\text{QCD}}^{\text{eff}} = \sum_f \left[\bar{q}_{L,f}i\slashed{D}q_{L,f} + \bar{q}_{R,f}i\slashed{D}q_{R,f} - m_f(\bar{q}_{L,f}q_{R,f} + \bar{q}_{R,f}q_{L,f})\right] - \frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu}. \tag{4.117}$$

Therefore, in the absence of a mass term, $\mathcal{L}$ is invariant under the chiral transformations

$$q_L = \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \to V_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = V_L q_L, \tag{4.118}$$

$$q_R \to V_R q_R, \tag{4.119}$$

where V_L and V_R are unitary 3×3 matrices. Let us stress that V_L and V_R are *global* rotations in flavor space (which leave colors unchanged), while the gauge transformations are local and act in color space. Instead of m_u, m_d, $m_s \to 0$, it is also useful to consider the two-flavor chiral limit $m_{u,d} \to 0$ and m_s fixed. In this case, the symmetry transformations are

$$\begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \to V_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}, \tag{4.120}$$

and the transformations can be parameterized as

$$V_{L,R} = \exp \left[i\alpha_{L,R} + i\frac{\sigma^a}{2} \alpha_{L,R}^a \right], \tag{4.121}$$

where the Pauli matrices $\frac{\sigma^a}{2}$, $a = 1, 2, 3$, are the generators of $SU(2)$. For the three-flavor case, the generators are the Gell-Mann matrices $\frac{\lambda^a}{2}$, $a = 1, \ldots, 8$.[4] We can then consider infinitesimal transformations and Noether's theorem gives a classically conserved current for each transformation $J_\mu \propto \frac{\delta \mathcal{L}}{\delta(\partial^\mu \psi)} \delta \psi$:

$$\begin{aligned} L_\mu &= \bar{q}_L \gamma_\mu q_L, & L_\mu^a &= \bar{q}_L \gamma^\mu \frac{\lambda^a}{2} q_L, \\ R_\mu &= \bar{q}_R \gamma_\mu q_R, & R_\mu^a &= \bar{q}_R \gamma^\mu \frac{\lambda^a}{2} q_R. \end{aligned} \tag{4.122}$$

Instead of left- and right-handed currents, it is convenient to use vector and axial-vector currents:

$$\begin{aligned} V^\mu &= L^\mu + R^\mu = \bar{q} \gamma^\mu q, \\ A^\mu &= R^\mu - L^\mu = \bar{q} \gamma^\mu \gamma^5 q. \end{aligned} \tag{4.123}$$

It turns out that A^μ is anomalous, i.e., $\partial_\mu A^\mu \neq 0$ due to quantum effects. Anomalies are a subtle effect in QFT, where certain classical symmetries are violated by quantum effects. We will not discuss this topic further in this lecture, but refer the reader to QFT textbooks such as [30, 31]. The anomaly of A^μ is also called the chiral anomaly or Adler–Bell–Jackiw anomaly [32, 33]. One can compute the divergence of the axial current from triangle Feynman diagrams and finds

$$\partial_\mu A^\mu = \frac{N_c g_s^2}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} G_a^{\mu\nu} G_a^{\rho\sigma}. \tag{4.124}$$

The non-conservation of the axial current has interesting phenomenological consequences, in particular, it explains why the lifetime of the neutral pion is much shorter than the one of the charged pions. We return to this point towards the end of our discussion of chiral perturbation theory.

The remaining $SU(3)_L \times SU(3)_R \times U(1)_V$ transformations are symmetries of the quantum theory. With each current, we can associate a conserved charge

$$Q_V^a = \int d^3x \, \bar{q} \gamma^0 \frac{\lambda^a}{2} q, \tag{4.125}$$

$$Q_A^a = \int d^3x \, \bar{q} \gamma^0 \gamma^5 \frac{\lambda^a}{2} q. \tag{4.126}$$

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[4]We stress again that the matrices λ^a introduced here act in flavor space not in color space.

4. The Standard Model at low energies

The $2 \times 8 + 1$ charges Q_V, Q_V^a, Q_A^a commute with the Hamiltonian $\mathbb{H}$ of massless QCD $[Q_V^a, \mathbb{H}] = [Q_A^a, \mathbb{H}] = 0$. The question is then whether the spectrum of the theory is symmetric, or whether the symmetry is spontaneously broken. Vafa and Witten have shown that the vector-like symmetries are unbroken $Q_V^a |0\rangle = 0$ [34]. For the axial-vector symmetry, the situation is more complicated. Let us discuss the two possibilities:

1. Unbroken symmetry $Q_A^a |0\rangle = 0$: in this case, the spectrum contains degenerate multiplets of $G = SU(3)_V \times SU(3)_A$.

2. Spontaneously broken symmetry $Q_A^a |0\rangle \neq 0$: in this case only multiplets of $SU(3)_V \subset G$ appear in the spectrum, and for each broken generator a Goldstone boson arises.

The second case is realized in nature. The statement that the spectrum contains a massless boson for each broken generator is Goldstone's theorem [38, 39]. A naive derivation of this theorem is obtained as follows. Since $[Q_A^a, \mathbb{H}] = 0$ we have

$$\mathbb{H} \, Q_A^a |0\rangle = Q_A^a \, \mathbb{H} |0\rangle = 0 \,. \tag{4.127}$$

but $Q_A^a |0\rangle \neq 0$, so there is a nontrivial state

$$Q_A^a |0\rangle$$

with zero energy for each broken generator, i.e. a massless particle. This state inherits the quantum numbers of Q_A^a so it is a parity-odd, spin-0 boson. Unfortunately, this simple argument has a flaw:

$$\langle 0| \, Q_A^a Q_A^b \, |0\rangle = \int d^3x \int d^3y \, \langle 0| \, A_0^a(x) A_0^b(y) \, |0\rangle \tag{4.128}$$

$$= \int d^3x \int d^3y \, F^{ab}(x - y) = \infty, \tag{4.129}$$

so the "states" $Q_A^a |0\rangle$ have infinite norm.

A rigorous proof of Golstone's theorem is obtained by analyzing the correlation function

$$\langle 0| \, [Q_A^a(t), P^b(t, \mathbf{y})] \, |0\rangle \,. \tag{4.130}$$

Inserting a basis of states and using current conservation, one can show that if this matrix element is non-vanishing, then the theory contains a massless particle with the same quantum numbers as the operators $P^a = \bar{q} \frac{\lambda^a}{2} \gamma_5 q$. A general proof of the Goldstone theorem based on the analysis of the above commutator is presented in the classic paper [40]. An Appendix with the derivation for our specific case can be found at [41].

For the specific case of symmetry breaking under consideration, the matrix element (4.130) can be simplified using the equal time anti-commutation relations

$$\{\psi_{\alpha,r}(t, \mathbf{x}), \psi_{\beta,s}^\dagger(t, \mathbf{y})\} = \delta_{\alpha\beta} \delta_{rs} \delta^{(3)}(\mathbf{x} - \mathbf{y}),$$
$$\{\psi_{\alpha,r}(t, \mathbf{x}), \psi_{\beta,s}(t, \mathbf{x})\} = 0 \,. \tag{4.131}$$

The commutator has the form

$$[ab, cd] = a\{b, c\}d - ac\{b, d\} + \{a, c\}db - c\{a, d\}b \,. \tag{4.132}$$

The matrix element (4.130) is nonzero only for $a = b$. Evaluating it for this case with a fixed value of a we get

$$[A_0^a(\mathbf{x}, 0), P^a(\mathbf{y}, 0)] = q^\dagger(y)\gamma^5\frac{\lambda^a}{2}\delta^{(3)}(\mathbf{x} - \mathbf{y})\gamma^0\gamma^5\frac{\lambda^a}{2}q(y) - q^\dagger(y)\gamma^0\gamma^5\frac{\lambda^a}{2}\delta^{(3)}(\mathbf{x} - \mathbf{y})\gamma^5\frac{\lambda^a}{2}q(y)$$

$$= -2\bar{q}\frac{(\lambda^a)^2}{4}q\delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{4.133}$$

Because of $SU(3)_V$ invariance of $|0\rangle$, one can average over the components,[5] leading to

$$\langle 0|\, [Q_A^a, P^a(x)]\, |0\rangle = -\frac{1}{8}\sum_b \frac{1}{2} \langle 0|\, \bar{q}(\lambda^b)^2 q\, |0\rangle \tag{4.134}$$

$$= -\frac{1}{3} \langle 0|\, \bar{q}q\, |0\rangle \tag{4.135}$$

$$= -\frac{1}{3} \langle 0|\, \bar{u}u + \bar{d}d + \bar{s}s\, |0\rangle \tag{4.136}$$

$$= - \langle 0|\, \bar{u}u\, |0\rangle\,, \tag{4.137}$$

where we used the fundamental Casimir operator $\sum_a(\lambda^a)^2 = 4C_F\mathbb{1} = \frac{16}{3}\mathbb{1}$. The quark condensate $\bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L$ breaks chiral symmetry.

A non-vanishing quark condensate implies that chiral symmetry is spontaneously broken and that there are 8 pseudoscalar Goldstone bosons. Since the quark masses are non-zero, chiral symmetry is not an exact symmetry of QCD. On the other hand, the u-, d-, and s-quark masses are small, so one can treat the mass term of QCD as a perturbation. Looking at the spectrum, one finds that three mesons $\pi^\pm$, π^0 are quite light, $M_\pi \approx 140\,\text{MeV}$, and nearly degenerate. Since they also are parity-odd and have spin zero, it is plausible that they are the $SU(2)$ triplet of "Goldstone" bosons associated with the spontaneous breaking of chiral symmetry in the $\begin{pmatrix} u \\ d \end{pmatrix}$ sector:

$$SU(2)_L \times SU(2)_R \to SU(2)_V. \tag{4.138}$$

Since the small mass-term breaks the symmetry explicitly, they acquire a small mass. For this reason, they are called *pseudo* Goldstone bosons. The lowest-lying eight mesons π^+, π^-, π^0, K^+, K^-, K^0, $\bar{K}^0$, and η have $J^P = 0^-$, and so match the pattern of symmetry breaking for $SU(3)_L \times SU(3)_R \to SU(3)_V$. If chiral symmetry were unbroken, one would expect multiplets of the full symmetry group: for each parity-odd meson, there should be a (nearly) degenerate parity-even partner. From these considerations, and from the fact that chiral perturbation theory is very successful in describing the low-energy phenomenology of QCD, one concludes that chiral symmetry is indeed spontaneously broken.

4.4.2. Transformation properties of Goldstone bosons

In order to construct the most general effective Lagrangian, we need to know how the Goldstone-boson fields $\boldsymbol{\pi}$ transform under chiral symmetry. Usually fields $\phi(x)$ transform linearly, as a representation of a symmetry group $\phi \to M(g)\phi$. Under such a transformation the classical vacuum $\phi(x) = 0$ automatically maps into itself. This is not appropriate for the

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[5]Formally, this can be derived by considering the analogous commutator $[Q_V^a, S^b]$.

case of spontaneously broken symmetries, where the vacuum transforms in a nontrivial way. Indeed, for Goldstone boson fields, the symmetry is realized non-linearly, as we will now see. More details on the Callan, Coleman, Wess, and Zumino (CCWZ) construction [42, 43] that underlies the following discussion is provided in Appendix C.

Let us consider first the general case of a group G that breaks spontaneously to a subgroup H. There are then $n = n_G - n_H$ Goldstone bosons, which we collect into an n-dimensional vector $\boldsymbol{\pi}(x)$. A realization of the group is a mapping

$$\boldsymbol{\pi} \to \boldsymbol{\pi}' = \boldsymbol{\varphi}(g, \boldsymbol{\pi}) \tag{4.139}$$

for any $g \in G$. This mapping must obey the composition law[6]

$$\boldsymbol{\varphi}(g_1, \boldsymbol{\varphi}(g_2, \boldsymbol{\pi})) = \boldsymbol{\varphi}(g_1 g_2, \boldsymbol{\pi}). \tag{4.140}$$

Remarkably, this property, together with the requirement that the unbroken subgroup should be realized linearly, determines $\boldsymbol{\varphi}$ uniquely. To see this, consider the image of the origin $\boldsymbol{\varphi}(g, \boldsymbol{\pi} = 0)$. The elements $h \in H$ map the origin onto itself $\boldsymbol{\varphi}(h, 0) = 0$, since H is linearly realized. Moreover

$$\boldsymbol{\varphi}(gh, 0) = \boldsymbol{\varphi}(g, 0) \quad \forall h \in H, \tag{4.141}$$

so that $\boldsymbol{\varphi}$ is defined on the coset space G/H. It maps an element of G/H into the space of Goldstone fields. Furthermore, the mapping is invertible since $\boldsymbol{\varphi}(g_1, 0) = \boldsymbol{\varphi}(g_2, 0)$ implies $g_1 H = g_2 H$. To see this, assume $\boldsymbol{\varphi}(g_1, 0) = \boldsymbol{\varphi}(g_2, 0)$ and consider the identity e of the group and note that

$$\boldsymbol{\varphi}(e, 0) = 0 = \boldsymbol{\varphi}(g_1^{-1}g_1, 0) = \boldsymbol{\varphi}(g_1^{-1}, \boldsymbol{\varphi}(g_1, 0)) = \boldsymbol{\varphi}(g_1^{-1}, \boldsymbol{\varphi}(g_2, 0)) = \boldsymbol{\varphi}(g_1^{-1}g_2, 0) = 0,$$

and therefore $g_1^{-1}g_2 \in H$, i.e., $g_1 H = g_2 H$, see Appendix C. Accordingly, the function $\boldsymbol{\varphi}(g, 0)$ provides a one-to-one mapping between the coset space G/H and the values of the $\boldsymbol{\pi}$ field. The transformation of the field follows from the action of $g \in G$ on the coset space. The only freedom left is the choice of coordinates on G/H.

Let us now consider $G = SU(2)_L \times SU(2)_R = \{(V_L, V_R)|V_L \in SU(2), V_R \in SU(2)\}$ and the unbroken vector subgroup $H = \{(V, V)|V \in SU(2)\}$. The coset space associated with an element g is the set $\tilde{g}H = \{(\tilde{V}_L V, \tilde{V}_R V)|V \in SU(2)\}$. To parameterize G/H, we select one element of each set $\tilde{g}H$. A possible choice is $U = \tilde{V}_R \tilde{V}_L^\dagger$, since

$$(\tilde{V}_L V, \tilde{V}_R V) = (1, \tilde{V}_R \tilde{V}_L^\dagger)\underbrace{(\tilde{V}_L V, \tilde{V}_L V)}_{\in H}. \tag{4.142}$$

The transformation law of U under G is

$$U \to V_R U V_L^\dagger \tag{4.143}$$

for $g = (V_L, V_R)$. In a final step we need to parameterize $U(x) \in SU(2)$. One can use the standard parameterization

$$U(x) = \exp\left[i\frac{\sigma^a \pi^a}{F}\right] = \exp\left[\frac{i}{F}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\right], \tag{4.144}$$

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[6]In general $\boldsymbol{\varphi}$ is not a representation, since it is not linear $\boldsymbol{\varphi}(g, \lambda\boldsymbol{\pi}) \neq \lambda\boldsymbol{\varphi}(g, \boldsymbol{\pi})$.

where we have rewritten the pion field in the linear combinations with definite electric charge. The factor F was introduced to obtain a dimensionless exponent, but it will correspond to the pion decay constant. One could have chosen a different parameterization, e.g., for the $SU(2)$ case often the co-called σ parameterization

$$U(x) = \sqrt{1 - \boldsymbol{\pi}^2/F^2} + \frac{i}{F}\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \tag{4.145}$$

proves beneficial. The π-fields of the two different parameterizations are related by a field redefinition, under which the physics remains unchanged. For $SU(3)$, the standard parameterization is

$$U(x) = \exp\left[\frac{i}{F}\lambda^a\pi^a\right] = \exp\left[\frac{i}{F}\begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}\right]. \tag{4.146}$$

To understand why the field are parameterized in this way, one needs to consider the quark-mass term and the coupling to photons, to which we will turn in the next subsection.

4.4.3. Effective Lagrangian

Now that we know the transformation properties of the Goldstone bosons, it is straightforward to write down the effective Lagrangian in the chiral limit $m_q = 0$. After this we will have to implement the symmetry-breaking terms involving the quark masses.

Under a chiral transformation $U \to V_R U V_L^\dagger$, so we need to find an effective Lagrangian $\mathcal{L}_{\text{eff}}(U)$ that is invariant under this transformation. Since $U(x)$ is dimensionless, the terms with higher orders of $U(x)$ are not suppressed, so instead we order terms by derivatives

$$\mathcal{L}_{\text{eff}} = f_0(U) + f_1(U)\square U + f_2(U)\partial_\mu U\partial^\mu U + \cdots, \tag{4.147}$$

where for now we have ignored the flavor indices, which will have to be contracted later. We observe:

1. Chiral symmetry implies $f_0(U) = f_0(V_R U V_L^\dagger)$. Choosing $V_R = \mathbb{1}$, $V_L = U$, this leads to $f_0(U) = f_0(\mathbb{1}) = \text{const}$. Therefore, terms of order $\mathcal{O}(1)$ in the derivative expansion only give an irrelevant constant that can be dropped.

2. The f_1-term can be absorbed into f_2 using integration by parts

$$\int d^4x f_1(U)\square U = -\int d^4x f_1'(U)\partial_\mu U\partial^\mu U, \tag{4.148}$$

3. We can rewrite the remaining term as

$$\mathcal{L}_{\text{eff}} = f(U)\partial_\mu U\partial^\mu U = \tilde{f}(U)\Delta_\mu\Delta^\mu \quad \text{with } \Delta_\mu = (\partial_\mu U)U^\dagger, \tag{4.149}$$

and the quantity Δ_μ transforms as $\Delta_\mu \to V_R\Delta_\mu V_R^\dagger$ and is invariant under V_L transformations. Choose now $V_L = U$ and $V_R = \mathbb{1}$ under which this term transforms to

$$\tilde{f}(U)\Delta_\mu\Delta^\mu \to \tilde{f}(UV_L^\dagger)\Delta_\mu\Delta^\mu = \tilde{f}(\mathbb{1})\Delta_\mu\Delta^\mu \tag{4.150}$$

4. The Standard Model at low energies

The last question is how the indices of the matrices Δ_μ are contracted. The only possibility to ensure invariance under V_R is

$$\mathcal{L}_{\text{eff}} = C \cdot \text{Tr}[\Delta_\mu \Delta^\mu] = C \cdot \text{Tr}[(\partial_\mu U)U^\dagger(\partial^\mu U)U^\dagger] \tag{4.151}$$

$$= -C \cdot \text{Tr}[\partial_\mu U U^\dagger U \partial^\mu U^\dagger] = -C \cdot \text{Tr}[\partial_\mu U \partial^\mu U^\dagger] \tag{4.152}$$

$$\equiv \frac{F^2}{4}\text{Tr}[\partial_\mu U \partial^\mu U^\dagger], \tag{4.153}$$

where the prefactor has been chosen to get canonically normalized kinetic terms for the pion fields. To see this, we now expand

$$U(x) = \exp\left[\frac{i}{F}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}\right] = \mathbb{1} + \frac{i}{F}\boldsymbol{\pi} \cdot \boldsymbol{\sigma} - \frac{1}{2F^2}\boldsymbol{\pi}^2 \mathbb{1} + \mathcal{O}(\pi^3), \tag{4.154}$$

leading to

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\left(-\frac{1}{F^2}\partial_\mu \pi^a \partial^\mu \pi^b\right)2\delta^{ab} + \mathcal{O}(\pi^3) = -\frac{1}{2}\partial_\mu \boldsymbol{\pi} \partial^\mu \boldsymbol{\pi} + \mathcal{O}(\pi^3). \tag{4.155}$$

The effective Lagrangian has several remarkable properties:

1. one parameter F determines all π-interactions,

2. symmetry requires interactions with arbitrary many pions,

3. derivative couplings: the interactions vanish if the momenta go to zero.

So far, our effective Lagrangian is only valid in the limit $m_q = 0$ and we should now also implement the quark-mass terms that break the symmetry:

$$\mathcal{L}_M = -\bar{q}_R M q_L - \bar{q}_L M^\dagger q_R, \tag{4.156}$$

with

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{4.157}$$

Note that $\mathcal{L}_M$ would be invariant if M transformed as $M \to V_R M V_L^\dagger$. This property can actually be used to construct $\mathcal{L}_{\text{eff}}(U, M)$: one treats M as an external source in the QCD Lagrangian (4.156) that transforms as $M \to V_R M V_L$ (a so-called "spurion" field). $\mathcal{L}_{\text{eff}}$ must then be invariant as well. Expanding in M, the lowest invariant term is

$$\mathcal{L}_{\text{symmetry breaking}} = \frac{F^2 B_0}{2}\text{Tr}[MU^\dagger + M^\dagger U]. \tag{4.158}$$

After one has constructed an invariant Lagrangian, one then inserts the actual matrix M in (4.157) into the Lagrangian. This term gives a mass to the pions, which becomes obvious after expanding in the fields. For $SU(2)$ one finds

$$\mathcal{L}_{\text{symmetry breaking}} = \frac{F^2 B_0}{2}\text{Tr}[M]\left(-\frac{1}{F^2}\boldsymbol{\pi}^2\right) = -\frac{B_0}{2}(m_u + m_d)\boldsymbol{\pi}^2 \equiv -\frac{M_\pi^2}{2}\boldsymbol{\pi}^2, \tag{4.159}$$

from which we conclude that the masses of the pions are equal and proportional to the sum $m_u + m_d$.

To relate the quantity B_0 to a QCD matrix element, we again treat M as an external source $M \equiv [M(x)]_{ij}$ and then take a functional derivative of the full and effective theory partition functions with respect to the source

$$\frac{1}{i}\frac{\delta}{\delta M_{ij}(x)}Z_{\text{QCD}} = -\langle 0|\,\bar{q}_{L,i}(x)q_{R,j}(x) + \bar{q}_{R,j}(x)q_{L,i}(x)\,|0\rangle\,, \tag{4.160}$$

$$\frac{1}{i}\frac{\delta}{\delta M_{ij}(x)}Z_{\text{eff}} = \frac{F^2 B_0}{2}\,\langle 0|\,(U^\dagger)_{ji}(x) + U_{ij}(x)\,|0\rangle\,. \tag{4.161}$$

The classical action is minimized by $\boldsymbol{\pi} = 0$, $U = \mathbb{1}$. Up to pion-loop corrections, we thus have

$$F^2 B_0 \delta_{ij} = -\langle 0|\,\bar{q}_{L,i}q_{R,j} + \bar{q}_{R,j}q_{L,i}\,|0\rangle\,, \tag{4.162}$$

$$F^2 B_0 = -\langle 0|\,\bar{u}u\,|0\rangle = -\langle 0|\,\bar{d}d\,|0\rangle\,. \tag{4.163}$$

This shows that B_0 corresponds to the quark condensate in the limit $m_q \to 0$. Taken together with the expansion of the pion mass, we find the relation

$$M_\pi^2 = \underbrace{(m_u + m_d)}_{\text{explicit breaking}} \underbrace{\left(\frac{-\langle 0|\,\bar{u}u\,|0\rangle}{F^2}\right)}_{\text{spontaneous breaking}} + \mathcal{O}\!\left(m_q^2\right), \tag{4.164}$$

known as the "Gell-Mann–Oakes–Renner relation" [35]. For $SU(2)$, the three pions have the same mass because the quadratic term in the expansion (4.154) is proportional to the unit matrix, essentially because $\frac{1}{2}\{\sigma^i, \sigma^j\} = \delta^{ij}\mathbb{1}$. In the $SU(3)$ case $\{\lambda^a, \lambda^b\}$ is non-trivial and one finds

$$\begin{aligned}
M_\pi^2 &= (m_u + m_d)B_0 + \mathcal{O}\!\left(m_q^2\right), \\
M_{K^\pm}^2 &= (m_u + m_s)B_0 + \mathcal{O}\!\left(m_q^2\right), \\
M_{K^0, \bar{K}^0}^2 &= (m_d + m_s)B_0 + \mathcal{O}\!\left(m_q^2\right), \\
M_\eta^2 &= \frac{1}{3}(m_u + m_d + 4m_s)B_0 + \mathcal{O}\!\left(m_q^2\right).
\end{aligned} \tag{4.165}$$

This explains why $M_K^2 \gg M_\pi^2$, because $m_s \gg m_u, m_d$. Combining the results (4.165) one obtains the Gell-Mann–Okubo mass formula [36, 37]

$$M_\pi^2 - 4M_K^2 + 3M_\eta^2 = \mathcal{O}\!\left(m_q^2\right). \tag{4.166}$$

To understand how the mesons interact with photons, W-, and Z-bosons, it is useful to introduce external sources with the appropriate quantum numbers both in the full and the effective theory. For QCD, we add

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_0 + \mathcal{L}_1, \tag{4.167}$$

$$\mathcal{L}_0 = -\frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu} + \bar{q}i\slashed{D}q, \tag{4.168}$$

$$\mathcal{L}_1 = v_\mu^a V_a^\mu + a_\mu^a A_a^\mu - s^a S_a - p^a P_a, \tag{4.169}$$

with sources

$$V_a^\mu = \bar{q}\gamma^\mu\frac{\lambda_q}{2}q, \qquad A_a^\mu = \bar{q}\gamma^\mu\gamma_5\frac{\lambda_q}{2}q, \qquad S_a = \bar{q}\frac{\lambda_q}{2}q, \qquad P_a = \bar{q}i\gamma_5\frac{\lambda_q}{2}q, \tag{4.170}$$

and one can also include singlet currents via $\lambda_0 = \sqrt{\frac{2}{3}}\mathbb{1}$. The external fields $v_\mu^a(x)$, $a_\mu^a(x)$, $s^a(x)$, $p^a(x)$ can be used to probe different aspects of QCD, e.g., quark masses are included in $s^a(x)$. To construct $\mathcal{L}_{\mathrm{eff}}$ in the presence of these sources, one can use the fact that $\mathcal{L}_{\mathrm{QCD}}$ becomes invariant under *local* transformations

$$q_L(x) \to V_L(x)q_L(x), \qquad q_R(x) \to V_R(x)q_R(x), \tag{4.171}$$

provided the external fields transform like gauge fields:

$$
\begin{aligned}
r_\mu &= v_\mu + a_\mu \to V_R(v_\mu + a_\mu)V_R^\dagger - i(\partial_\mu V_R)V_R^\dagger, \\
l_\mu &= v_\mu - a_\mu \to V_L(v_\mu - a_\mu)V_L^\dagger - i(\partial_\mu V_L)V_L^\dagger, \\
s + ip &\to V_R(s + ip)V_L^\dagger,
\end{aligned} \tag{4.172}
$$

where $v_\mu = v_\mu^a \frac{\lambda^a}{2}$, etc., and the task is then to construct a locally invariant effective Lagrangian. At leading order, it is sufficient to replace ∂_μ by the covariant derivative:

$$iD_\mu U = i\partial_\mu U + (v_\mu + a_\mu)U - U(v_\mu - a_\mu), \tag{4.173}$$

where v_μ, a_μ count as $\mathcal{O}(p)$, so that

$$\mathcal{L}_{\mathrm{eff}} = \frac{F^2}{4}\mathrm{Tr}[D_\mu U D^\mu U^\dagger] + \frac{F^2 B_0}{2}\mathrm{Tr}[\chi U^\dagger + \chi^\dagger U] + \mathcal{O}(p^4), \tag{4.174}$$

with $\chi = s + ip$ and the convention $D_\mu U^\dagger \equiv (D_\mu U)^\dagger$. We are now in the position to interpret the second free parameter F. Let us considere the axial-vector current. In the SM, the matrix element of the axial current is responsible for the pion decay, see (4.81). To compute the matrix element of the axial current in the effective theory, we expand

$$D_\mu U = \frac{i}{F}\boldsymbol{\sigma} \cdot \partial_\mu \boldsymbol{\pi} - 2i\boldsymbol{a}_\mu \cdot \frac{\boldsymbol{\sigma}}{2} + \mathcal{O}(\boldsymbol{\pi}^2). \tag{4.175}$$

For the coupling of a single axial-vector current we thus find

$$\frac{F^2}{4}\frac{4}{F}\left(-\boldsymbol{a}_\mu \cdot \partial^\mu \boldsymbol{\pi}\right) = -F\boldsymbol{a}_\mu \cdot \partial^\mu \boldsymbol{\pi}, \tag{4.176}$$

Matching to (4.81) shows that $F = F_\pi$ at leading order.

At $\mathcal{O}(p^4)$ $\mathcal{L}_{\mathrm{eff}}$ has the form [2,3]

$$
\begin{aligned}
\mathcal{L}^{(4)} = &\frac{l_1}{4}(\mathrm{Tr}[D_\mu U D^\mu U^\dagger])^2 + \frac{l_2}{4}\mathrm{Tr}[D_\mu U D_\nu U^\dagger] \times \mathrm{Tr}[D^\mu U D^\nu U^\dagger] \\
&+ \frac{l_3}{4}(\mathrm{Tr}[\chi U^\dagger + U\chi^\dagger])^2 + \frac{l_4}{4}\mathrm{Tr}[D_\mu \chi D^\mu U^\dagger + D_\mu U D^\mu \chi^\dagger] + \cdots
\end{aligned} \tag{4.177}
$$

For $SU(3)$ $\mathcal{L}^{(4)}$ has 12 coupling constants, while for $SU(2)$ 10 such low-energy constants arise.

To perform calculations beyond leading order, one needs one-loop graphs from $\mathcal{L}^{(2)}$, which also count as $\mathcal{O}(p^4)$, e.g.,

$$\underset{\mathcal{L}^{(2)}}{\underline{\bigcirc}} \propto \int d^4 k \frac{1}{k^2 - M_\pi^2}k^2 \propto M_\pi^4 \tag{4.178}$$

contributes to the $\mathcal{O}(m_q^2)$ corrections in the mass formulae (4.165). In particular, loop contributions are suppressed by a loop factor $1/(16\pi^2)$ and, for dimensional reasons, by $1/F_\pi^2$, leading to an expansion in $1/(4\pi F_\pi)^2$.[7] From the loop suppression one thus expects an expansion in

$$\left\{ \frac{p^2}{\Lambda_\chi^2}, \frac{M_\pi^2}{\Lambda_\chi^2}, \frac{M_K^2}{\Lambda_\chi^2} \right\}, \tag{4.179}$$

where M_π and M_K apply to the $SU(2)$ and $SU(3)$ expansions, respectively. The RG estimate for the scale of chiral symmetry breaking can be compared to the first resonance in the spectrum, which gives rise to

$$0.775\,\text{GeV} \approx M_\rho \lesssim \Lambda_\chi \lesssim 4\pi F_\pi \approx 1.2\,\text{GeV}, \tag{4.180}$$

and thus a scale around 1 GeV. In practice, the expansion proceeds not just in terms of powers of Eq. (4.179), but the loop corrections can generate logarithmic dependences on momenta and masses.

Finally, we mention a complication that appears in the construction of chiral Lagrangians, which is related to the Wess–Zumino–Witten (WZW) anomaly [45, 46]. The point is that $\mathcal{L}_{\text{QCD}}$ and $\mathcal{L}_{\text{eff}}$ are invariant under local chiral transformations, but the partition function

$$Z[\nu, a, s, p] = \int \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{D}A_\mu\, e^{i\int d^4x(\mathcal{L}_0 + \mathcal{L}_1)} = e^{iS_{\text{eff}}[\nu, a, s, p]} \tag{4.181}$$

is not invariant if the external sources are non-zero, because of anomalies in the fermion determinant. Since the effective theory does not involve fermion fields, invariance of $\mathcal{L}_{\text{eff}}$ leads to invariance of the partition function. To correct for this mismatch, one needs to add $\mathcal{L}_{\text{eff}}$ a term that reproduces the charge of the QCD partition function. This term is called the WZW term $\mathcal{L}_{\text{WZW}}$. The full effective theory Lagrangian is then

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{WZW}}. \tag{4.182}$$

The WZW term is $\mathcal{O}(p^4)$ and does not involve any low-energy constants. In contrast to $\mathcal{L}_{\text{inv}}$, the terms in $\mathcal{L}_{\text{WZW}}$ contain odd numbers of Goldstone-boson fields. In particular, it contains a term describing an interaction of two vector fields with a π^0, which leads to

$$\Gamma_{\pi^0 \to \gamma\gamma} = \frac{\alpha^2 N_c^2 M_{\pi^0}^3}{64\pi^3 F_\pi^2} \left(Q_u^2 - Q_d^2\right)^2 = 7.749(15)\,\text{eV}. \tag{4.183}$$

The good agreement with the experimental value $7.802(117)\,\text{eV}$ [47] is sometimes sold as evidence for $N_c = 3$. However, Bär and Wiese pointed out that $Q_u^2 - Q_d^2 = \frac{1}{N_c}$ for N_c colors (to ensure anomaly cancellation in the SM), so that the rate does not depend on N_c [48]. The π^0 decay is much faster than the one of the charged pion because it is mediated by strong/electromagnetic instead of electroweak interactions.

4.5. The Standard Model as an EFT

We have now covered (almost[8]) all sectors of the SM and discussed the corresponding EFTs. In particular, we have worked our way up in energy and have integrated out heavy leptons,

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[7]This argument can be made more rigorous in terms of the renormalization group [44].

[8]Heavy-quark EFT requires a non-relativistic formalism, see Chapter 5.

quarks, and gauge bosons. In each case we wrote down the relevant operators up to $d = 6$. In all cases, the main motivation for the EFT approach was either to simplify calculations in the SM or, as for strong interactions at low energies, to actually make analytic calculations possible.

Going one step further, one could also consider the entire SM as an EFT, i.e., assume that all potential new particle arise above the scale of electroweak symmetry breaking and can thus be described by effective operators that obey the SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. While there is only a single operator at $d = 5$ (related to neutrino Majorana masses), there are many possibilities at $d = 6$, and finding the minimal set is non-trivial. A complete set was first written down by Buchmüller and Wyler [49]. However, the *minimal* set was only constructed in 2010 [50], comprising $15 + 19 + 25 = 59$ different operators (bosonic, two-fermion, and four-fermion, respectively). If baryon number is violated, four additional operators appear.
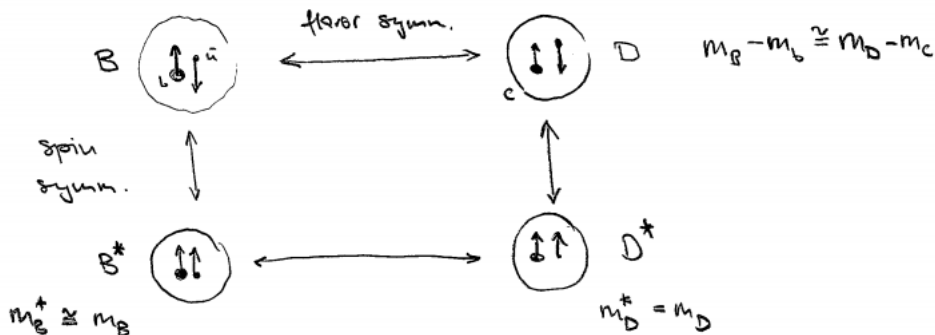
5. Non-relativistic effective theories

We have considered several EFTs that are obtained by integrating out heavy particles. However, in many cases heavy particles are present even at very low energy. The reason are conservation laws for particle numbers such as lepton number conservation ($L = L_{e^-} - L_{e^+}$) and baryon (or quark) number conservation. If we neglect the weak interaction, then each lepton and quark flavor is separately conserved.

The proper framework to describe heavy particles at low momentum are non-relativistic EFTs. Examples of systems that can be studied with such techniques are atoms, mesons with heavy (i.e., bottom or charm) quarks, and protons interacting with slow pions, etc.

A heavy B-meson has similarities to a hydrogen atom, but an important difference is that the light degrees of freedom inside the B-meson are still highly relativistic and strongly interacting. Nevertheless some properties of hydrogen carry over: the energy of the B-meson is to good accuracy independent of the b-quark spin. Also, the energy spectrum of B-mesons is independent of the heavy-quark mass to good approximation:

Heavy quark effective theory (HQET) will allow us to derive such relations in the limit $m_Q \to \infty$ and to systematically analyze the $\frac{1}{m_Q}$ corrections. For systems such as hydrogen

5. Non-relativistic effective theories

or a B_c meson also lighter fermions (e^- or $\bar{c}$ respectively) can be treated non-relativistically. The EFT for this case is non-relativistic QED/QCD (NRQED/NRQCD). HQET and NRQCD have the same Lagrangian but different power counting.

5.1. Heavy-quark effective theory

Interactions of the heavy quark Q with the light constituents of a heavy-to-light meson will change its momentum by amounts of order $\Lambda_{\text{QCD}} \sim 1$ GeV, but its velocity is barely changed $\delta v_Q^\mu = \frac{\Delta p_Q^\mu}{m_Q} \ll 1$. To analyze such systems, we introduce a reference vector v^μ, $v^2 = 1$, in the direction of the heavy quark and split

$$p_Q^\mu = m_Q v^\mu + r^\mu, \tag{5.1}$$

so that the residual momentum r^μ is $\mathcal{O}(\Lambda_{\text{QCD}})$. A popular choice for v^μ is the meson velocity $v^\mu = \frac{P_M^\mu}{m_M}$. The EFT then corresponds to an expansion in the residual momentum r^μ over the heavy quark mass m_Q.

On the level of the quark field the decomposition of the momentum is achieved by splitting off the large phase $e^{-im_Q v \cdot x}$ from the field:

$$\psi_Q(x) = e^{-im_Q v \cdot x}\{h_v(x) + H_v(x)\}, \tag{5.2}$$

where

$$h_v(x) = e^{im_Q v \cdot x} P_+ \psi_Q(x), \qquad P_+ = \frac{\mathbb{1} + \slashed{v}}{2}, \tag{5.3}$$

$$H_v(x) = e^{im_Q v \cdot x} P_- \psi_Q(x), \qquad P_- = \frac{\mathbb{1} - \slashed{v}}{2}. \tag{5.4}$$

The projection operators P_+ and P_- split the field into the large ("upper") components $h_v(x)$ and the small ("lower") components $H_v(x)$. They obey

$$\slashed{v} h_v(x) = h_v(x), \qquad \slashed{v} H_v(x) = -H_v(x). \tag{5.5}$$

Let us insert this decomposition into the Dirac Lagrangian:

$$\mathcal{L}_Q = \bar{\psi}_Q(i\slashed{D} - m_Q)\psi_Q \tag{5.6}$$

$$= \bar{h}_v i\slashed{D} h_v + \bar{H}_v(i\slashed{D} - 2m_Q)H_v + \bar{H}_v i\slashed{D} h_v + \bar{h}_v i\slashed{D} H_v, \tag{5.7}$$

which further simplifies to

$$\mathcal{L}_Q = \bar{h}_v iv \cdot D h_v + \bar{H}_v(-iv \cdot D - 2m_Q)H_v + \bar{H}_v i\slashed{D}_\perp h_v + \bar{h}_v i\slashed{D}_\perp H_v, \tag{5.8}$$

when expressed in terms of the component $D_\perp^\mu = D^\mu - v \cdot D v^\mu$ perpendicular to v^μ. Here we used that

$$\bar{h}_v \gamma^\mu h_v = \bar{h}_v \gamma^\mu \slashed{v} h_v = -\bar{h}_v \gamma^\mu h_v + 2v^\mu \bar{h}_v h_v = v^\mu \bar{h}_v h_v, \tag{5.9}$$

$$\bar{H}_v \gamma^\mu H_v = -v^\mu \bar{H}_v H_v, \tag{5.10}$$

$$\bar{H}_v \slashed{v} h_v = \bar{h}_v \slashed{v} H_v = 0. \tag{5.11}$$

The EOM for H_v is

$$(-iv \cdot D - 2m_Q)H_v + i\slashed{D}_\perp h_v = 0, \tag{5.12}$$

which can be formally inverted as

$$H_v = \frac{1}{2m_Q} \sum_{n=0}^{\infty} \left(-\frac{iv \cdot D}{2m_Q}\right)^n i\slashed{D}_\perp h_v. \tag{5.13}$$

This shows that H_v is suppressed with respect to h_v by a factor $\frac{r_\perp}{2m_Q}$ and can be integrated out. Since the action is quadratic in the fields, this step can even be done exactly. At the classical level, the result is obtained by inserting the solution of the EOM for H_v back into $\mathcal{L}_Q$. At leading order one obtains

$$\mathcal{L}_Q = \bar{h}_v iv \cdot Dh_v + \frac{1}{2m_Q} \underbrace{\bar{h}_v i\slashed{D}_\perp i\slashed{D}_\perp h_v}_{\text{power corrections}} + \mathcal{O}\left(\frac{1}{m_Q^2}\right). \tag{5.14}$$

The power corrections can be further rewritten as

$$i\slashed{D}_\perp i\slashed{D}_\perp = iD_\mu^\perp iD_\nu^\perp \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\right) \tag{5.15}$$

$$= iD_\mu^\perp iD_\nu^\perp (g_{\mu\nu} - i\sigma^{\mu\nu}) \tag{5.16}$$

$$= (iD_\perp)^2 + \frac{i}{2}[D_\mu^\perp, D_\nu^\perp]\sigma^{\mu\nu} \tag{5.17}$$

$$= (iD_\perp)^2 + \frac{g_s}{2}\sigma^{\mu\nu}G_{\mu\nu}, \tag{5.18}$$

where we used the analog of Eq. (4.3) for the QCD field strength tensor.[1] The resulting Lagrangian

$$\mathcal{L}_Q = \bar{h}_v iv \cdot Dh_v + \frac{1}{2m_Q}\bar{h}_v(iD_\perp)^2 h_v + \frac{g_s}{4m_Q}\bar{h}_v\sigma_{\mu\nu}G^{\mu\nu}h_v \tag{5.19}$$

simplifies further when going to the rest frame $v^\mu = (1, \mathbf{0})$

$$\mathcal{L}_Q = \bar{h}_v iD_t h_v + \frac{1}{2m_Q}\bar{h}_v \mathbf{D}^2 h_v - \frac{g_s}{2m_Q}\bar{h}_v \boldsymbol{\sigma} \cdot \mathbf{B}_c h_v, \tag{5.20}$$

where $\mathbf{B}_c$ is the chromomagnetic field and the notation has been changed to two-component spinors. The first term is independent of the quark spin ("heavy-quark spin symmetry") and the quark mass ("heavy-quark flavor symmetry"). The second term breaks heavy-quark flavor symmetry but maintains the spin symmetry, while the third term violates both.

5.2. Connection to quantum mechanics

Let us go into the rest frame of the heavy quark $v^\mu = (1, \mathbf{0})$. The projection operator is then

$$P_+ = \frac{1}{2}(\mathbb{1} + \gamma_0) = \begin{pmatrix} \mathbb{1} & \\ & 0 \end{pmatrix}, \tag{5.21}$$

---

[1]The corrections from $D_\mu^\perp$ vs. D_μ are symmetric in $\mu \leftrightarrow \nu$ and thus vanish upon contraction with $\sigma^{\mu\nu}$.

i.e., P_+ projects out the upper two components of the Dirac field. When considering QED instead of QCD,

$$\psi_Q \to \psi_e, \qquad iD_\mu \to i\partial_\mu - eA_\mu,$$

the magnetic operator becomes

$$-\frac{e}{4m_e}\bar{h}_v\sigma^{\mu\nu}F_{\mu\nu}h_v = -\frac{ie}{8m_e}\bar{h}_v\underbrace{\left(\sigma^i\sigma^j - \sigma^j\sigma^i\right)}_{2i\varepsilon^{ijk}\sigma^k}\underbrace{\left(\partial_i A_j - \partial_j A_i\right)}_{2\partial_i A_j}h_v \tag{5.22}$$

$$= \frac{e}{2m_e}\bar{h}_v\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\mathbf{A})h_v = \frac{e}{2m_e}\bar{h}_v\boldsymbol{\sigma}\cdot\mathbf{B}h_v. \tag{5.23}$$

The effective Lagrangian for a slow electron (described by a field X) is therefore

$$\mathcal{L} = \bar{X}iD_tX - \bar{X}\frac{(i\boldsymbol{D})^2}{2m_e}X + \frac{e}{2m_e}\bar{X}\boldsymbol{\sigma}\cdot\mathbf{B}X. \tag{5.24}$$

The EOM associated with this Lagrangian is the Schrödinger equation for an e^- interacting with a photon field. The free propagator associated with $\mathcal{L}$
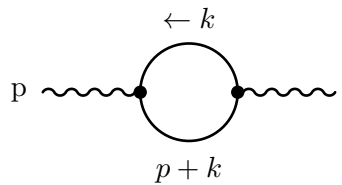
$$\triangle_x = \frac{1}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon} \tag{5.25}$$

has only a *single* pole, in contrast to a relativistic propagator:

$$\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{2\omega}\left[\underbrace{\frac{1}{p^0 - \omega + i\varepsilon}}_{\text{particle}} - \underbrace{\frac{1}{p^0 + \omega - i\varepsilon}}_{\text{antiparticle}}\right] \tag{5.26}$$

$$= \frac{1}{2m}\frac{1}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon} + \dots, \tag{5.27}$$

where $\omega = m + \frac{\mathbf{p}^2}{2m} + \dots$, $p^0 = m + E$. This has important consequences: since the theory no longer contains anti-particles, closed fermion loops vanish:

$$\text{p} \sim\!\!\sim\!\!\sim \bigcirc \sim\!\!\sim\!\!\sim \propto \int d^dk \frac{1}{(E + k^0) - \frac{(\mathbf{p+k})^2}{2m} + i\epsilon}\frac{1}{k^0 - \frac{\mathbf{k}^2}{2m} + i\epsilon} = 0, \tag{5.28}$$

because we can choose the k^0 integration contour without encountering a pole (since Im $k^0 < 0$ for all poles). Therefore all fermion loops vanish in HQET. The effect of virtual anti-particles can be absorbed into the Wilson coefficients of the operators in $\mathcal{L}_{\text{eff}}$, e.g.,
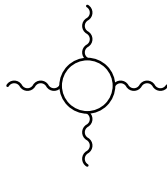
is represented by the Euler–Heisenberg terms in $\mathcal{L}_{\text{eff}}$.

Despite this observation, our theory is not simply quantum mechanics, since the electromagnetic field is a fully relativistic quantum field. To obtain QM, we need to treat also the electromagnetic field as a classical one. Let us therefore assume that $A^\mu = \phi(0, \mathbf{x})$ is a fixed classical potential (e.g., the Coulomb field of a proton).

Now the field operator fulfills

$$i\partial_t \hat{X} = [\hat{X}, \hat{\mathbb{H}}] = \left(-\frac{\boldsymbol{\nabla}^2}{2m} + e\phi(\mathbf{x}) \right) \hat{X}. \tag{5.29}$$

The solutions of the time-independent Schrödinger equation

$$\left(-\frac{\boldsymbol{\nabla}^2}{2m} + e\phi(\mathbf{x}) \right) \varphi_n(x) = E_n \varphi_n(x) \tag{5.30}$$

form a complete set of functions, which can be used to expand

$$\hat{X}(t, \mathbf{x}) = \sum_i \hat{a}_i e^{-iE_i t} \varphi_i(x). \tag{5.31}$$

The operator $\hat{a}_i$ annihilates the state with associated wave function $\varphi_i(x)$. Now the system is indeed quantum mechanical: the one-particle states are $|i\rangle = a_j^\dagger |0\rangle$ and they have associated wave functions

$$\langle 0| \hat{X}(t, \mathbf{x}) |i\rangle = e^{-iE_i t} \varphi_i(x) \tag{5.32}$$

that fulfill the Schrödinger equation.

To summarize

1. The EFT for a non-relativistic particle has a Lagrangian that has the Schrödinger equation as EOM.

2. There are no anti-particles in the EFT, their effect can be absorbed into the Wilson coefficients, since they are highly virtual.

3. Treating the photon as a classical background field we recover quantum mechanics.

5.3. Non-relativistic QED and QCD

Let us finally consider the effective theory relevant for the description of bound states of two heavy particles, e.g., positronium (e^+e^-), muonium ($\mu^+\mu^-$), bottomonium ($\bar{b}b$), and charmonium ($\bar{c}c$). The effective theories are called NRQED and NRQCD, respectively, and are closely related to HQET, except for the fact that we now deal with both a particle, described by a two-component spinor ψ, and an anti-particle, which we denote by χ. The effective Lagrangian has the form

$$\mathcal{L}_{\text{NR}} = \mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{\text{mixed}} + \mathcal{L}_{\text{light}}. \tag{5.33}$$

$\mathcal{L}_{\text{mixed}}$ contains operators involving both χ and ψ fields. $\mathcal{L}_{\text{light}}$ is the QCD Lagrangian for the light quarks plus higher-dimensional operators. The Lagrangians for the ψ field is nothing

5. Non-relativistic effective theories

but the HQET Lagrangian evaluated for $v^\mu = (1, \mathbf{0})$, since it is natural to work in the rest frame of the bound state. Therefore,

$$\mathcal{L}_\psi = \psi^\dagger \left(iD_t + \frac{\boldsymbol{D}^2}{2m_\psi} \right)\psi + \frac{1}{8m_\psi^3}\psi^\dagger \boldsymbol{D}^4 \psi + \frac{gC_1}{2m_\psi}\psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{B}\psi + \frac{gC_2}{8m_\psi^2}\psi^\dagger \left(\boldsymbol{D} \cdot \mathbf{E} - \mathbf{E} \cdot \boldsymbol{D} \right)\psi$$
$$+ \frac{gC_3}{8m_\psi^2}\psi^\dagger \left(i\boldsymbol{D} \times \mathbf{E} - \mathbf{E} \times i\boldsymbol{D} \right) \cdot \boldsymbol{\sigma}\psi, \tag{5.34}$$

where g is the gauge coupling, $g = g_s$ or $g = -e$. Moreover, we have included $1/m_Q^2$- and $1/m_Q^3$-suppressed terms because the power counting is different than in HQET. Next, $\mathcal{L}_\chi$ is obtained as the charge conjugate of $\mathcal{L}_\psi$,

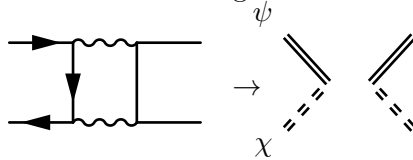$$\mathcal{L}_\chi = \mathcal{L}_\psi \big|_{\psi \to \chi,\, A^\mu \to -A^\mu}, \tag{5.35}$$

and the lowest-dimensional operators in $\mathcal{L}_{\text{mixed}}$ are four-quark operators, e.g.,

$$\mathcal{L}_{\text{mixed}} = \frac{C_4}{m^2}\psi^\dagger \psi \, \chi^\dagger \chi + \frac{C_5}{m^2}\psi^\dagger \boldsymbol{\sigma}\sigma_2 \chi \, \chi^\dagger \sigma_2 \boldsymbol{\sigma}\psi. \tag{5.36}$$

The first operator arises when high-energy contributions to the scattering of χ and ψ are integrated out, e.g.,
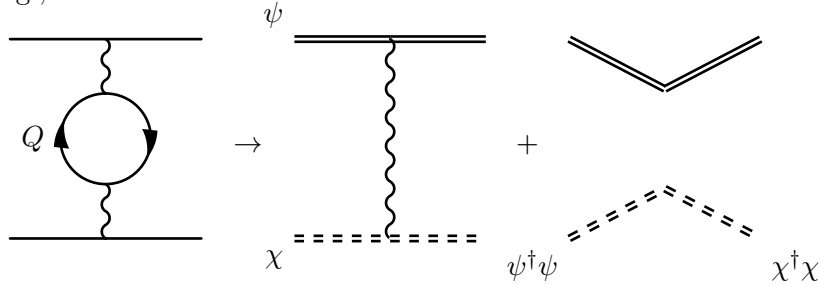
The second operator arises in annihilation diagrams such as

These diagrams only exist if χ is the anti-particle of ψ. They have an imaginary part, which describes the decay $\psi\chi \to \gamma\gamma$ (or gg), and accordingly C_5 is imaginary. The effective $\mathbb{H}$ is not Hermitian and the theory is not unitary! However, there is a good physical reason for this violation of unitarity: bound states, such as e^+e^-, decay over time. The imaginary part of $\mathbb{H}$ encodes the decay rate. The probability for finding the e^- in e^+e^- is not 1 for all times, because it will annihilate sooner or later.

This is the first complication compared to HQET. The second one is that the static Lagrangian

$$\mathcal{L} = \psi^\dagger i\partial_t \psi \tag{5.37}$$

cannot serve as a starting point in non-relativistic theories. There are formal arguments to show this, but the simple physical reason is that the e^+e^- in the bound state are not static. They are close to their mass shell $E = \frac{\mathbf{p}^2}{2m} + \dots$ and we thus should count $D_t \sim \frac{\boldsymbol{D}^2}{2m} \sim \frac{m\mathbf{v}^2}{2}$ as of the same order. Instead of powers of $1/m_Q$, we should count powers of $v = |\mathbf{v}|$. A third

complication is that the multiple photon/gluon exchanges between ψ and χ are unsuppressed. More precisely, the exchange of Coulomb gluons needs to be taken into account to all orders. In Coulomb gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, the gauge Lagrangian reads

$$
\begin{aligned}
-\frac{1}{4} G^{\mu\nu} G_{\mu\nu} &= \frac{1}{2} G^{0i} G_{0i} - \frac{1}{4} G^{ij} G_{ij} \\
&= \frac{1}{2} \left[(\partial_i A_0)^2 + (\partial_0 A_i)^2 - (\partial_i A_j)^2 + \text{``non-Abelian'' terms} \right].
\end{aligned} \tag{5.38}
$$

The field A_0 has no time derivatives and is thus not propagating. Since the action is quadratic, one can integrate out A^0. Its effect is then described by a potential, which is just the Fourier transform of its propagator

$$
V(\mathbf{x} - \mathbf{y}) = g_s^2 \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \frac{1}{\mathbf{k}^2} = \frac{g_s^2}{4\pi |\mathbf{x} - \mathbf{y}|}. \tag{5.39}
$$

The leading-order effective Lagrangian for a non-relativistic particle–antiparticle pair is then

$$
\mathcal{L}_{\mathrm{NR}} = \int d^3 x \, \psi^\dagger \left[i\partial_t + \frac{\boldsymbol{\nabla}^2}{2m} \right] \psi - \int d^3 x_1 \int d^3 x_2 \psi^\dagger(x_1) t^a \psi(x_1) \chi^\dagger(x_2) t^a \chi(x_2) V(\mathbf{x}_1 - \mathbf{x}_2). \tag{5.40}
$$

Accounting for $V(\mathbf{x})$ to all orders amounts to solving the Schrödinger equation. The remaining terms are treated as perturbations. Unfortunately, we thus found that the problem involves three different scales

	m (hard)	mv (soft)	mv^2 (ultrasoft)
$e^+ e^-$ value	0.5 MeV	3.7 keV	25 eV

where we have used that for positronium $V \sim \alpha$. These three scales make it difficult to organize the computations. In particular, in dimensional regularization the non-relativistic integrals receive contributions from the hard region, since the scale m_Q appears in the integrand. Initially people used to perform the computations with a hard cutoff, which avoids this problem but makes computations extremely cumbersome. Using the threshold expansion [51], which is also called the "strategy of regions," it became possible to eliminate the unwanted hard corrections in dimensional regularization and to separate the soft and ultrasoft corrections. An EFT approach that implements this separation on the level of the Lagrangian is "velocity NRQCD" or "vNRQCD," first proposed in Ref. [52]. An earlier solution called "potential NRQCD" ("pNRQCD") [53] amounts to integrating out the soft scale mv and to constructing an effective theory containing only ultrasoft degrees of freedom:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{QCD}} \qquad &(\mu > m) \qquad &\text{hard + soft + ultrasoft} \\
\downarrow \qquad & & \\
\mathcal{L}_{\mathrm{NRQCD}} \qquad &(m > \mu > mv) \qquad &\text{soft + ultrasoft} \\
\downarrow \qquad & & \\
\mathcal{L}_{\mathrm{pNRQCD}} \qquad &(mv > \mu > mv^2) \qquad &\text{ultrasoft}
\end{aligned}
$$

The fields in pNRQCD are not quarks and anti-quarks, but color-singlet and color-octet $\bar{Q}Q$-pairs:

$$
S \equiv S(r) \sim \chi^\dagger(-\frac{r}{2}) \psi(\frac{r}{2}), \quad O^a \sim \chi^\dagger t^a \psi, \tag{5.41}
$$

where S is the singlet and O the octet pair, which interact through potentials $V(r)$ and ultrasoft gluons.

A. Loop integrals in dimensional regularization

In this appendix we derive the standard formula

$$I(\alpha, \beta, \Delta) = \int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^\alpha}{(k^2 - \Delta + i\epsilon)^\beta} = \frac{i(-1)^{\alpha+\beta}}{(4\pi)^{d/2}} \frac{\Gamma\left(\alpha + \frac{d}{2}\right) \Gamma\left(\beta - \alpha - \frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{}{\Gamma(\beta)} \Delta^{\alpha - \beta + d/2}, \quad (A.1)$$

and show that all one-loop integrals can be brought into this form.

As a first step, we consider $I(\alpha, \beta, \Delta)$ for parameters $\alpha + d/2 > 0$ (IR convergence) and $\beta - \alpha - d/2 > 0$ (UV convergence), by performing a Wick rotation to Euclidean space $k^0 = ik_E^0$, $\mathbf{k} = \mathbf{k}_E$. Dropping the Euclidean label in the following, this gives

$$I(\alpha, \beta, \Delta) = i(-1)^{\alpha+\beta} \int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^\alpha}{(k^2 + \Delta)^\beta} = \frac{i(-1)^{\alpha+\beta}\Omega_d}{(2\pi)^d} \int_0^\infty dk \frac{k^{d-1+2\alpha}}{(k^2 + \Delta)^\beta}$$

$$= \frac{i(-1)^{\alpha+\beta}\Omega_d}{2(2\pi)^d} \Delta^{\alpha-\beta+d/2} \int_0^\infty dx \frac{x^{\alpha-1+d/2}}{(1+x)^\beta}, \qquad (A.2)$$

where we used polar coordinates in d dimensions with area Ω_d of the unit sphere und changed the integration to $x = k^2/\Delta$. The remaining integral can be brought into standard form by the transformation $y = x/(1+x)$, with $dx = dy/(1-y)^2$,

$$I(\alpha, \beta, \Delta) = \frac{i(-1)^{\alpha+\beta}\Omega_d}{2(2\pi)^d} \Delta^{\alpha-\beta+d/2} \int_0^1 dy\, y^{\alpha+d/2-1} (1-y)^{\beta-\alpha-d/2-1}$$

$$= \frac{i(-1)^{\alpha+\beta}\Omega_d}{2(2\pi)^d} \Delta^{\alpha-\beta+d/2} \frac{\Gamma\left(\alpha + \frac{d}{2}\right) \Gamma\left(\beta - \alpha - \frac{d}{2}\right)}{\Gamma(\beta)}, \qquad (A.3)$$

where in the last step we applied the general relation for the Beta function

$$\mathrm{B}(a,b) = \int_0^1 dy\, y^{a-1}(1-y)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \qquad (A.4)$$

The final result (A.1) then follows with $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$, which can be derived from Gaussian integrals in d dimensions

$$\int d^d x\, e^{-\mathbf{x}^2} = \left[\int dx\, e^{-x^2}\right]^d = \pi^{d/2}$$

$$= \Omega_d \int_0^\infty dx\, x^{d-1} e^{-x^2} = \frac{\Omega_d}{2} \int_0^\infty dy\, y^{d/2-1} e^{-y} = \frac{\Omega_d}{2} \Gamma\left(\frac{d}{2}\right). \qquad (A.5)$$

The derivation only applies for integer values of d, with the general result *defined* by

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}. \qquad (A.6)$$

Similarly, our derivation of (A.1) only applies as long as the integral is IR and UV convergent. However, the right-hand side is an analytic function of d, α, and β except for poles for $\alpha + d/2$ and $\beta - \alpha - d/2$ at $0, -1, -2, \ldots$. The integral is then defined by the analytic continuation in these variables. To evaluate the limit of $d = 4 - 2\epsilon \to 4$, one often needs the expansion

$$\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left(\frac{1}{\epsilon} - \gamma_E + 1 + \cdots + \frac{1}{n} \right) + \mathcal{O}(\epsilon), \tag{A.7}$$

where $\gamma_E = 0.5772\ldots$ is the Euler–Mascheroni constant.

To demonstrate that all one-loop diagrams indeed take the form (A.1) one uses Feynman parameterizations to combine multiple propagators into a single one, in the simplest case using

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1 - x)B]^2}. \tag{A.8}$$

A non-trivial example including Lorentz indices is given by

$$\begin{aligned}
S^{\mu\nu} &= \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 + i\epsilon)((k - p)^2 + i\epsilon)} = \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{k^\mu k^\nu}{[k^2 - 2xp \cdot k + xp^2 + i\epsilon]^2} \\
&= \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{k^\mu k^\nu}{[(k - xp)^2 + x(1 - x)p^2 + i\epsilon]^2}.
\end{aligned} \tag{A.9}$$

Shifting $k \to k + xp$, we can identify $\Delta = -x(1 - x)p^2 - i\epsilon$ and find

$$\begin{aligned}
S^{\mu\nu} &= \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{k^\mu k^\nu + x^2 p^\mu p^\nu + x(p^\mu k^\nu + k^\mu p^\nu)}{(k^2 - \Delta)^2} \\
&= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{\frac{g^{\mu\nu}}{d} k^2 + x^2 p^\mu p^\nu}{(k^2 - \Delta)^2},
\end{aligned} \tag{A.10}$$

where we used that the linear terms in k vanish upon integration. For the quadratic terms we used Lorentz invariance, i.e.,

$$\int \frac{d^d k}{(2\pi)^d} k^\mu k^\nu f(k^2) = g^{\mu\nu} \int \frac{d^d k}{(2\pi)^d} \tilde{f}(k^2). \tag{A.11}$$

Upon contraction with $g^{\mu\nu}$ this gives

$$\int \frac{d^d k}{(2\pi)^d} k^2 f(k^2) = d \int \frac{d^d k}{(2\pi)^d} \tilde{f}(k^2) \tag{A.12}$$

and thus $\tilde{f}(k^2) = f(k^2) k^2 / d$ in the integral. In the form (A.10) the master formula (A.1) applies.

For more complicated integrals the generalized Feynman parameterization reads

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \frac{\Gamma(m)}{\Gamma(m_1) \cdots \Gamma(m_n)} \int_0^1 dx_1 \cdots \int_0^1 dx_n \, \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{\prod_{i=1}^n x_i^{m_i - 1}}{\left[\sum_{i=1}^n x_i A_i\right]^m}, \tag{A.13}$$

where $m = \sum_{i=1}^n m_i$. To derive this relation, we proceed by induction in n. The case $n = 2$ follows by taking derivatives of

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \, \delta(1 - x - y) \frac{1}{(xA + yB)^2} \tag{A.14}$$

A. Loop integrals in dimensional regularization

with respect to A and B, which gives

$$\frac{1}{A^{m_1} B^{m_2}} = \frac{\Gamma(m_1 + m_2)}{\Gamma(m_1)\Gamma(m_2)} \int_0^1 dx \int_0^1 dy\, \delta(1 - x - y) \frac{x^{m_1-1} y^{m_2-1}}{(xA + yB)^{m_1+m_2}}. \tag{A.15}$$

In the induction step, we assume (A.13) for $n - 1$, and then use (A.15) to combine A_n with the rest:

$$
\begin{aligned}
\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} &= \frac{\Gamma(m)}{\Gamma(m - m_n)\Gamma(m_n)} \int_0^1 dx \int_0^1 dy\, \delta(1 - x - y) x^{m_n-1} y^{m-m_n-1} \\
&\quad \times \frac{\Gamma(m - m_n)}{\Gamma(m_1) \cdots \Gamma(m_{n-1})} \int_0^1 dx_1 \cdots \int_0^1 dx_{n-1} \frac{\delta\left(1 - \sum_{i=1}^{n-1} x_i\right) \prod_{i=1}^{n-1} x_i^{m_i-1}}{\left[xA_n + y\left(\sum_{i=1}^{n-1} x_i A_i\right)\right]^m} \\
&= \frac{\Gamma(m)}{\Gamma(m_1) \cdots \Gamma(m_n)} \int_0^1 dy\, y^{m-m_n-1} \int_0^1 dx_1 \cdots \int_0^1 dx_n\, \delta(1 - y - x_n) \\
&\quad \times \delta\left(1 - \sum_{i=1}^{n-1} x_i\right) \frac{\prod_{i=1}^{n} x_i^{m_i-1}}{\left[x_n A_n + y\left(\sum_{i=1}^{n-1} x_i A_i\right)\right]^m}. \tag{A.16}
\end{aligned}
$$

The induction step follows by rescaling $x_i \to x_i/y$ for $i = 1, \ldots, n - 1$ and the remaining integral

$$
\int_0^1 dy\, y^{-1} \delta(1 - y - x_n)\, \delta\left(1 - \frac{1}{y}\sum_{i=1}^{n-1} x_i\right) = \int_0^1 dy\, y^{-1} \delta(1 - y - x_n)\, y\delta\left(y - \sum_{i=1}^{n-1} x_i\right)
$$

$$
= \delta\left(1 - \sum_{i=1}^{n} x_i\right). \tag{A.17}
$$

The upper integration boundaries are not affected and can be set to ∞ in intermediate steps to simplify the argument.

B. Feynman rules for derivative couplings

Feynman rules are usually formulated in momentum space. For theories involving derivative couplings they are obtained after Fourier transforming the action to momentum space, just as for the standard case without derivatives. In Fourier space, a derivative ∂_μ acting on an incoming particle with momentum p gives $-ip_\mu$ and the opposite sign for an outgoing particle. To derive the Feynman rules, it is simplest to consider all momenta to be incoming. Moreover, to facilitate calculations it is useful to symmetrize the Feynman rules in the external legs.

Let us consider as an example the interaction term

$$\delta\mathcal{L} = \frac{g}{4!}\phi^2(x)\Box\phi^2(x) \tag{B.1}$$

and transform it to momentum space according to

$$\phi(x) = \int_k e^{-ik\cdot x}\tilde{\phi}(k) = \int \frac{d^dk}{(2\pi)^d}e^{-ik\cdot x}\tilde{\phi}(k), \tag{B.2}$$

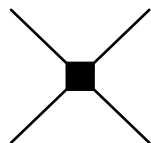where k is an incoming momentum. In Fourier space the interaction becomes

$$\int d^dx\,\delta\mathcal{L} = \int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\int d^dx\,\frac{g}{4!}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\big[-(k_3+k_4)^2\big]\tilde{\phi}(k_3)\tilde{\phi}(k_4)e^{-i(k_1+k_2+k_3+k_4)\cdot x}$$

$$= -\frac{g}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\tilde{\phi}(k_1)\tilde{\phi}(k_2)(k_3+k_4)^2\tilde{\phi}(k_3)\tilde{\phi}(k_4)(2\pi)^d\delta^{(4)}(k_1+k_2+k_3+k_4)\,, \tag{B.3}$$

where the δ-function arose from the integration over x. The disadvantage of this form is that two of the fields are singled out, so that in the application one would need to remember on which fields the derivatives act. It is much easier to symmetrize the Feynman rule using momentum conservation, i.e.,

$$\int d^dx\,\delta\mathcal{L} = -\frac{g}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)(2\pi)^d\delta^{(4)}(k_1+k_2+k_3+k_4)$$

$$\times \frac{1}{3}\big[(k_1+k_2)^2+(k_1+k_3)^2+(k_1+k_4)^2\big], \tag{B.4}$$

which is now completely symmetric in the three distinct momentum pairs. The associated Feynman rule is

$$= -i\frac{g}{3}\big[(k_1+k_2)^2+(k_1+k_3)^2+(k_1+k_4)^2\big]\,. \tag{B.5}$$

The 4! is canceled by the combinatorial factor from all possible Wick contractions, just as for ϕ^4 theory.

B. Feynman rules for derivative couplings

Let us consider the contribution of this interaction term to the four-point function, i.e., the scattering process

$$\phi(p_1)\phi(p_2) \rightarrow \phi(p_3)\phi(p_4), \tag{B.6}$$

with $p_{1,2}$ incoming and $p_{3,4}$ outgoing. Accordingly, the amplitude becomes

$$\mathcal{M} = -\frac{g}{3}\left[(p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2\right] = -\frac{g}{3}\left[p_1^2 + p_2^2 + p_3^2 + p_4^2\right], \tag{B.7}$$

which on the mass shell, $p_i^2 = m^2$, reduces to $\mathcal{M} = -4gm^2/3$.

C. CCWZ construction of phenomenological Lagrangians

The construction of effective Lagrangians becomes more complicated if the symmetries of the full theory are not fully realized by the ground state, i.e., if a symmetry becomes spontaneously broken. In this case, a method to construct the effective Lagrangian was developed by Callan, Coleman, Wess, and Zumino (CCWZ) in Refs. [42, 43]. Here, we follow the presentation from Ref. [54].

Suppose, the full theory is invariant under the group G, while the ground state is only invariant under the subgroup H of G, giving rise to $n = n_G - n_H$ Goldstone bosons, where n_G and n_H denote the number of generators. The Goldstone bosons are described by fields ϕ_i, collected in a vector $\Phi = (\phi_1, \ldots, \phi_n)$. This defines a vector space

$$M_1 \equiv \left\{ \Phi : M^4 \to \mathbb{R}^n | \phi_i : M^4 \to \mathbb{R} \right\}, \tag{C.1}$$

with M^4 Minkowski space. The main point in the CCWZ construction amounts to establishing a connection between the so-called quotient G/H and the Goldstone-boson fields, in such a way that the effective Lagrangian can then be parameterized by resorting to a set of variables parameterizing the elements of G/H. The application described in a main text concerns low-energy QCD with $G = SU(3)_L \times SU(3)_R$ and $H = SU(3)_V$ ("Chiral perturbation theory"), another generalized realizations of the Higgs sector with $G = SU(2)_L \times U(1)_Y$ and $H = U(1)_{\text{EM}}$ ("Higgs effective field theory").

The aim is to find a mapping $\phi(g, \Phi)$ from $G \times M_1 \to M_1$ with the following properties

$$\phi(e, \Phi) = \Phi \quad \forall \, \Phi \in M_1, \quad e \text{ identity of } G,$$
$$\phi(g_1, \phi(g_2, \Phi)) = \phi(g_1 g_2, \Phi) \quad \forall \, g_1, g_2 \in G, \, \Phi \in M_1. \tag{C.2}$$

Such a mapping is called an *operation* of G on M_1, and the second condition the *group-homomorphism property*. Note that we do not require this mapping to be linear, i.e., in general $\phi(g, \lambda \Phi) \neq \lambda \phi(g, \Phi)$, so the result will *not* define a representation.

Let us first consider $\Phi = 0$, for which all fields are mapped onto the origin in $\mathbb{R}^n$, which, in a theory with Goldstone boson only, can be interpreted as the ground-state configuration. Since the ground state is invariant under H, we require that $\phi(h, 0) = 0$ for $h \in H$. Next, we turn to the quotient G/H, which is defined as the set of all left cosets $\{gH | g \in G\}$. Here, the set $gH = \{gh | h \in H\}$ defines the left coset of $g \in G$, and the quotient is the set of all such cosets. An important property of this construction is that cosets either completely overlap or are completely disjoint.

Before proceeding, let us illustrate these properties using the symmetry group C_4 of a square with directed sides:

C. CCWZ construction of phenomenological Lagrangians
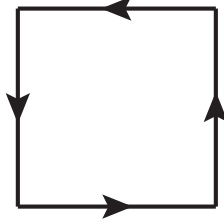
This group consists of four elements $C_4 = \{e, a, a^2, a^3\}$, where a can be interpreted as a rotation by $\pi/2$. The nontrivial subgroup is $H = \{e, a^2\}$, with left cosets

$$eH = \{e, a^2\} = a^2 H, \qquad aH = \{a, a^3\} = a^3 H. \tag{C.3}$$

The quotient G/H therefore consists of the two elements $\{e, a^2\}$ and $\{a, a^3\}$. Since the elements of G/H are completely disjoint, any element of a given coset uniquely represents the coset in which it appears. It is this property that is exploited in the CCWZ construction, to which we now return.

For $g \in G$ and $h \in H$ we have

$$\phi(gh, 0) = \phi(g, \phi(h, 0)) = \phi(g, 0), \tag{C.4}$$

i.e., the action on $\Phi = 0$ is identical among a given coset gH, which can be interpreted in such a way that the origin is mapped onto the same vector in $\mathbb{R}^n$. Second, the mapping is injective with respect to the elements of G/H (no two elements are mapped onto the same Φ), which can be seen as follows: consider $g, g' \in G$ with $g' \notin gH$ and let us assume $\phi(g, 0) = \phi(g', 0)$. Then

$$0 = \phi(e, 0) = \phi(g^{-1}g, 0) = \phi(g^{-1}, \phi(g, 0)) = \phi(g^{-1}, \phi(g', 0)) = \phi(g^{-1}g', 0), \tag{C.5}$$

which implies $g^{-1}g' \in H$, i.e., $g' \in gH$, in contradiction to the assumption, so that $\phi(g, 0) = \phi(g', 0)$ must be false. From this one concludes that there exists an *isomorphic mapping* between G/H and the Goldstone-boson fields. To account for the fact that they also depend on $x \in M^4$ (and are not just constant vectors in $\mathbb{R}^n$ as assumed so far), the cosets gH are also allowed to depend on x.

For the construction of the effective Lagrangian we need the transformation behavior of the Goldstone-boson fields under $g \in G$, which we can now study based on the isomorphism just established. To each Φ corresponds a coset $\tilde{g}H$ with some $\tilde{g}$. Let $f = \tilde{g}h \in \tilde{g}H$ denote a representative of this coset, i.e.,

$$\Phi = \phi(f, 0) = \phi(\tilde{g}h, 0). \tag{C.6}$$

Now,

$$\phi(g, \Phi) = \phi(g, \phi(\tilde{g}h, 0)) = \phi(g\tilde{g}h, 0) = \phi(f', 0) = \Phi', \qquad f' \in g(\tilde{g}H), \tag{C.7}$$

so, in order to obtain the transformed Φ' from a given Φ we simply need to multiply the left coset $\tilde{g}H$ representing Φ by g in order to obtain the new left coset representing Φ'. This procedure then determines the transformation behavior of the Goldstone bosons, leaving the task of finding a convenient parameterization of G/H.

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