Effective field theories — Exercise 6

Method of regions, SCET — 10.12.2015

1. Consider two light-cone vectors n_μ and $\bar{n}_\mu$, with $\bar{n} \cdot n = 2$. Show that the operators

$$P_+ = \frac{\not{n}\not{\bar{n}}}{4}\,, \qquad\qquad P_- = \frac{\not{\bar{n}}\not{n}}{4}\,,$$

are projection operators with $P_+ + P_- = 1$.

2. Consider the QED Wilson line

$$[z, y] = \exp\left[-ie \int_{\mathcal{C}} dx_\mu A^\mu(x)\right]\,,$$

where the curve $\mathcal{C}$ goes from y to z.

(a) Show that under a gauge transformation $V(x) = \exp(i\alpha(x))$ the Wilson line transforms as

$$[z, y] \to V(z)[z, y]V^\dagger(y)\,.$$

(b) Show that the covariant derivative along the curve $\mathcal{C}$ vanishes, i.e.

$$\dot{x}^\mu(t) D_\mu\,[x(t), y] = 0\,,$$

where $D_\mu = i\partial_\mu + ieA_\mu(x)$ is the derivative with respect to x^μ and $x^\mu(t)$ a parameterization of the curve $\mathcal{C}$.

3. Using the method of regions, verify that the leading term in the expansion of the integral discussed in the lecture is given by

$$I = \int_0^\infty dk\, \frac{k}{(k^2 + m^2)\,(k^2 + M^2)} = -\frac{1}{M^2}\left[\ln\frac{m}{M} + \mathcal{O}\left(\frac{m^2}{M^2}\right)\right]\,.$$

The following integral is useful

$$\int_0^\infty dx\, \frac{x^a}{(1+x)^b} = \frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)}\,.$$

4. Compute the leading term in the expansion of the integral

$$I(a) = \int_0^\infty dt \, \frac{\sin(t)}{(t+a)^2}$$

for small a. To do so, introduce a factor t^ϵ into the integrand and use the method of regions. You might want to use Mathematica for the integrations; using it, it also easy check whether your result is correct.

5. Compute the soft loop integral

$$I_s = i\pi^{-d/2}\mu^{4-d} \int d^d k \, \frac{1}{k^2(n \cdot k \, \bar{n} \cdot p + p^2)(\bar{n} \cdot k \, n \cdot l + l^2)} \, .$$

For progagators linear in k such as $b = n \cdot k \, \bar{n} \cdot p + p^2$, it is useful to work with a modified Feynman parameterizations such as

$$\frac{1}{ab} = \int_0^\infty d\eta \, \frac{1}{(a+\eta b)^2} \, , \qquad \frac{1}{abc} = \int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \, \frac{2}{(a+\eta_1 b + \eta_2 c)^3} \, .$$