1. Consider two light-cone vectors $n_\mu$ and $\bar{n}_\mu$, with $\bar{n} \cdot n = 2$. Show that the operators

$$P_+ = \frac{\bar{n} \cdot k}{4}, \quad P_- = \frac{n \cdot k}{4},$$

are projection operators with $P_+ + P_- = 1$.

2. Consider the QED Wilson line

$$[z, y] = \exp\left[ -ie \int_C dx_\mu A_\mu(x) \right],$$

where the curve $C$ goes from $y$ to $z$.

(a) Show that under a gauge transformation $V(x) = \exp(i\alpha(x))$ the Wilson line transforms as

$$[z, y] \rightarrow V(z)[z, y] V^\dag(y).$$

(b) Show that the covariant derivative along the curve $C$ vanishes, i.e.

$$\dot{x}^\mu(t) D_\mu [x(t), y] = 0,$$

where $D_\mu = i\partial_\mu + ieA_\mu(x)$ is the derivative with respect to $x^\mu$ and $x^\mu(t)$ a parameterization of the curve $C$.

3. Using the method of regions, verify that the leading term in the expansion of the integral discussed in the lecture is given by

$$I = \int_0^\infty dk \frac{k}{(k^2 + m^2)(k^2 + M^2)} = -\frac{1}{M^2} \left[ \ln \frac{m}{M} + \mathcal{O}\left( \frac{m^2}{M^2} \right) \right].$$

The following integral is useful

$$\int_0^\infty dx \frac{x^a}{(1+x)^b} = \frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)}.$$
4. Compute the leading term in the expansion of the integral

\[ I(a) = \int_0^\infty dt \frac{\sin(t)}{(t + a)^2} \]

for small \( a \). To do so, introduce a factor \( t^\epsilon \) into the integrand and use the method of regions. You might want to use Mathematica for the integrations; using it, it also easy check whether your result is correct.

5. Compute the soft loop integral

\[ I_s = i\pi^{-d/2} \mu^{4-d} \int \frac{d^d k}{k^2(n \cdot k \bar{n} \cdot p + p^2)(\bar{n} \cdot k n \cdot l + l^2)} \]

For propagators linear in \( k \) such as \( b = n \cdot k \bar{n} \cdot p + p^2 \), it is useful to work with a modified Feynman parameterizations such as

\[
\begin{align*}
\frac{1}{ab} &= \int_0^\infty d\eta \frac{1}{(a + \eta b)^2}, & \frac{1}{abc} &= \int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \frac{2}{(a + \eta_1 b + \eta_2 c)^3}.
\end{align*}
\]