Effective field theories — Exercise 5

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1. Derive the Gordon identity ($q = p_2 - p_1$)

$$\bar{u}(p_2)\,\gamma^\mu\,u(p_1) = \frac{1}{2m}\,\bar{u}(p_2)\,\left[(p_1+p_2)^\mu + i\sigma^{\mu\nu}q_\nu\right]\,u(p_1)\,.$$

2. Consider a free scalar field theory for a particle of mass μ. Compute the causal propagator (the Pauli-Jordan function) by expanding the field in creation and annihilation operators

$$i\Delta(x-y) = \langle 0|\,[\phi(x),\phi^\dagger(y)]\,|0\rangle = \int d^4p\,\delta(p^2-\mu^2)\,\theta(p^0)\left[e^{-ip(x-y)} - e^{ip(x-y)}\right]$$

Reminder:

$$\phi(x) = \int \frac{d^3p}{\sqrt{2E_p}(2\pi)^{3/2}}\left[a_p e^{-ipx} + a_p^\dagger e^{ipx}\right]$$

with

$$[a_p, a_{p'}^\dagger] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}') \qquad\qquad [a_p, a_{p'}] = 0\,.$$

3. Spectral representation of the causal propagator. Show that the propagator in an interacting theory can be written as an integral over a spectral density $\rho(\mu^2)$:

$$\langle 0|\,[\phi(x),\phi^\dagger(y)]\,|0\rangle = i\int d\mu^2\,\rho(\mu^2)\,\Delta(x-y)\,,$$

where $\Delta(x-y)$ is the free-theory function derived in the previous exercise. This can be achieved by writing out the commutator and inserting a complete set of states, see Appendix D of the lecture notes for inspiration.

4. Go through the steps of the derivation of the spectral representation of the current correlator in Appendix D of the lecture notes.

5. In a theory with a single complex scalar field and spontaneous symmetry breaking (due to the standard Mexican hat potential), it is convenient to parameterise the excitations of the field around the vaccuum as

$$\Phi(x) = \frac{1}{\sqrt{2}}(v + h(x))e^{i\phi(x)} \,. \tag{1}$$

Here $h(x)$ is the massive Higgs boson, while $\phi(x)$ is the Goldstone boson. The invariance of the theory under $U(1)$ transformations implies that the Lagrangian must be invariant under a shift $\phi(x) \to \phi(x) + \alpha$. Note that $\phi(x)$ is dimensionless.

(a) Write down the general form of $\mathcal{L}_{\text{eff}}(\phi)$, the effective Lagrangian of the Goldstone boson. Include terms with two or less derivatives on the field. Note that the Lagrangian must be shift-invariant.

(b) Write also the terms with up to four derivatives on the fields.