Effective field theories — Exercise 1

Path integral — 18.9.2015

The course web site is http://www.becher.itp.unibe.ch/eft/.

There is a forum for questions on the ILIAS page of the course
https://ilias.unibe.ch/goto_ilias3_unibe_crs_877363.html.

1. Prove that in Quantum Mechanics

$$\begin{aligned}&\langle 0|\boldsymbol{T}\big(\hat{X}(t_1)\hat{X}(t_2)\ldots\hat{X}(t_n)\big)|0\rangle\\ &= \lim_{T\to(1-i\epsilon)\infty}\frac{1}{Z}\int\mathcal{D}X(t)\exp\left(i\int_{-T}^{T}dt\,\mathcal{L}(X(t),\dot{X}(t))\right)X(t_1)X(t_2)\ldots X(t_n)\,,\end{aligned}$$

where Z is the same path integral without position arguments $X(t_1)\ldots X(t_n)$. The Heisenberg position operator is denoted by $\hat{X}(t)$.

To derive the relation, start with the path-integral expression for the Schrödinger kernel

$$\langle X_f,\,t_f|X_i,\,t_i\rangle = \int\mathcal{D}X(t)\exp\left(i\int_{t_i}^{t_f}dt\,\mathcal{L}(X(t),\dot{X}(t))\right)\,, \tag{1}$$

which you know from other lectures and the theoretical exercises.

The exercise has two parts:

(a) Show that inserting position arguments $X(t_1)X(t_2)\ldots X(t_n)$ into the path integral (1) yields the time-ordered expectation values of the corresponding operators $\hat{X}(t_1)\hat{X}(t_2)\ldots\hat{X}(t_n)$. The completeness relation

$$\int dX|X,t\rangle\langle X,t| = 1\,,$$

is helpful, as always in path-integral derivations.

(b) Consider the limit

$$\lim_{T\to-(1-i\epsilon)\infty}|X_i,\,t_i\rangle \tag{2}$$

and show that the small imaginary part of the time argument suppresses the contribution of higher states so that the ground state $|0\rangle$ starts to dominate in the limit. To do so, rewrite the Heisenberg basis vector $|X_i,\,t_i\rangle$ in terms of time independent Schrödinger position basis vectors $|X\rangle$ and insert a complete set of energy eigenstates.

2. Read in your favourite QFT text book the section about path integrals for scalar fields (the basics are good enough), which contains a derivation of the relation we derived in QM in the previous exercise. Two useful free resources on the topic are

   (a) http://isites.harvard.edu/fs/docs/icb.topic473482.files/13-pathintegral.pdf

   (b) http://www.desy.de/~jlouis/Vorlesungen/QFTII11/QFTIIscript.pdf

   Note: The first reference fails to distinguish the field eigenstate $|\phi\rangle$ with eigenvalue 0 from the vacuum state $|0\rangle$. (The position eigenstate $|X\rangle$ with eigenvalue 0 is not the same as the vacuum state $|0\rangle$!) The second reference means $1 - i\epsilon$ when it writes $1 - \epsilon$.

   Please use the ILIAS page if you would like me to discuss a particular apsect during the lectures.