D. Goldstone's theorem

Assume that we have a theory which is invariant under a symmetry group $G$. This leads to a set of conserved currents. Consider one such current $J^A(x)$ and the associated conserved charge

$$Q^A = \int d^3 x \ J^A(x).$$

Let's assume that $Q^A |\Omega\rangle \neq 0$, i.e., that the symmetry is spontaneously broken.

Let's assume that $H |\Omega\rangle = 0 |\Omega\rangle$. It then follows that

$$H \phi^A |\Omega\rangle = [H, \phi^A] |\Omega\rangle + Q^A H |\Omega\rangle = 0.$$
We have a state $Q^A 10\rangle$ which is degenerate in energy with the vacuum.

Problem: $\|Q^A 10\rangle\|^2 = \langle 0 1 | Q^A Q^A | 10 \rangle$

$$= \int dx \int dy \, \langle 0 1 | Q^A (x) Q^A (y) | 10 \rangle = \infty$$

$F(x-y)$ (translation inv.)

This by itself is only moderately disturbing, since

$$\langle \vec{p}' | \vec{p} \rangle = (2\pi)^3 \omega_p \delta^{(2)}(\vec{p} - \vec{p}')$$

$\infty$ for $\vec{p} = \vec{p}'$

but the state $Q^A 10\rangle$ is also strange since it does not carry momentum. (What is a massless particle without 3-momentum?)
A clean way to identify particles is to look at the Fourier transform of the two-point function of two local operators. A $\frac{1}{p^2}$ pole indicates the presence of massless particles.

Let's assume that we have a set of scalar fields $\phi_a$ with the right quantum numbers of the Goldstone bosons (GBs) so that

$$[Q^A, \phi_a] = (t^A)_{ab} \phi_b$$

For a fundamental scalar field, this follows from

$$j^A = i \frac{\partial L}{\partial (\partial_\mu \phi_a)} t^A_{ab} \phi_b$$

and the canonical commutation relations, but our field $\phi_a$ can be a general, composite object.
We further assume that
\[ \langle 0 \mid \phi_a(0) \mid 0 \rangle \neq 0 \] for some \( a \).

and will now show that this implies the presence of massless bosons.

Following Gobletz, Solom and Weinberg, we analyze

\[
\langle 0 \mid \left[ J^a_\mu(y), \phi_a(x) \right] \mid 0 \rangle
= e^{i p_x \phi(0)} e^{-i p_x \phi(0)}
\]

\[
= \sum_x \left[ \langle 0 \mid J^a_\mu(y) \mid x \rangle \langle x \mid \phi(x) \mid 0 \rangle - \langle 0 \mid \phi(x) \mid x \rangle \langle x \mid J^a_\mu(y) \mid 0 \rangle \right]
\]

\[
= \sum_x \left[ e^{i p_x (x - y)} \langle 0 \mid J^a_\mu(0) \mid x \rangle \langle x \mid \phi(0) \mid 0 \rangle - e^{-i p_x (x - y)} \langle 0 \mid \phi(0) \mid x \rangle \langle x \mid J^a_\mu(0) \mid 0 \rangle \right]
\]

\[
= \sum_p \sum_x \delta^{(d)}(p - p_x) \left[ \ldots \right]
\]
\[
\sum_x d^4(p-x) \langle 01 | J^A_{\mu}(0) | x \rangle \langle x | b_{\alpha}(0) | 10 \rangle \\
= ip_{\mu}(p^0) \rho_{\alpha}^A(p^2) \\
\uparrow \\
\text{spectral density}
\]

\[
\sum_x d^4(p-x) \langle 01 | \phi_{\alpha}(0) | x \rangle \langle x | J^A_{\mu}(0) | 10 \rangle \\
= ip_{\mu}(p^0) \tilde{\rho}_{\alpha}^A(p^2)
\]

For \( x^0 = y^0 \) and \( |x-y| > 0 \) (spacelike separation), the commutator should vanish, which implies

\[
\rho_{\alpha}^A(p^2) = -\tilde{\rho}_{\alpha}^A(p^2). \quad [\text{To see it, do } \vec{p} \rightarrow -\vec{p}\text{ in the second term}].
\]

\[
\langle 01 | \left[ J^A_{\mu_1,\nu_1}(x) \right] | 10 \rangle = -\frac{2}{\Xi_{\mu_1}} \int d^4p \Theta(p^0) \\
\left[ e^{ip(x-y)} \rho_{\alpha}^A(p^2) + e^{-ip(x-y)} \tilde{\rho}_{\alpha}^A(p^2) \right] \\
\left[ e^{ip(x-y)} - e^{-ip(x-y)} \right] \rho_{\alpha}^A(p^2)
\]
By introducing $\lambda = \int \mu^2 \, d(p^2 - \mu^2)$, we finally have the following spectral representation

$$\langle 0 | \hat{J}_\mu \lambda \phi_{\alpha}(x) \phi_{\alpha}(y) \rangle = -\frac{\partial}{\partial y^\mu} \int \mu^2 \, \rho^A_{\alpha}(\mu^2) \, \Delta(x-y)$$

with

$$\Delta(z) = \int \rho \, \delta(p^2 - \mu^2) \, \Theta(p^0) \left[ e^{ipz} - e^{-ipz} \right]$$

the causal propagator of a free, massive scalar field.

$$\Delta(z) = 0 \quad \text{for} \quad z^2 < 0$$
$$\neq 0 \quad \text{for} \quad z^2 > 0$$

Now we use that $\Theta(p^0) = 0$:

$$0 = -\Box_y \int \mu^2 \, \rho^A_{\alpha}(\mu^2) \, \Delta(x-y)$$
$$= \int \mu^2 \, \rho^A_{\alpha}(\mu^2) \, \mu^2 \, \Delta(x-y)$$
\[ \Delta \equiv \text{Im} \Delta = -p^2 \text{Im} \Delta \]

Since \( \Delta(z) \neq 0 \) for \( z^2 > 0 \), we must have

\[ \rho^A_0 (\mu^2) \mu^2 = 0 \quad (\rho^A_0 (\mu^2) \neq 0!) \]

\( \rho^A_0 (\mu^2) \) is a distribution, so this relation implies

\[ \rho^A_0 (\mu^2) = c_0^A \cdot \delta(\mu^2) \]

If \( c \neq 0 \), the correlation function contains states with mass \( \mu^2 = 0 \) ! To fix \( c \), consider

\[ x^0 = y^0 = 0 \quad \text{and} \quad \]

\[ \langle \langle f^A \bar{J}_p^A (y), f^A (x) \rangle \rangle \]

\[ = 2i \int d^2p \rho^A_0 (\mu^2) \mathcal{S}d^3p \phi^A (p^0) \delta (p^2 - \mu^2) \rho^0 e^{i\bar{p}(x-y)} \]

\[ = i \int d^2p \rho^A_0 (\mu^2) \mathcal{S}d^3p \frac{e^{i\bar{p}(x-y)}}{(2\pi)^3 \delta (x-y)} \]
\[ \int \delta^{3}y \langle 0 \left| [Q_{\alpha}^{A}(y), \phi_{\alpha}(x)] \right| 10 \rangle \]

\[ = \langle 01 \left| Q_{\alpha}^{A}, \phi_{\alpha}(x) \right| 10 \rangle \]

\[ = i(2\pi)^{3} \int \delta \mu^{2} \rho_{\alpha}(\mu^{2}) = i(2\pi)^{3} C_{a}^{A} \]

But we assumed

\[ [Q_{\alpha}^{A}, \phi_{\beta}] = t_{ab}^{A} \phi_{\beta} \]

\[ \rightarrow i(2\pi)^{3} C_{a}^{A} = t_{ab}^{A} \langle 01 \phi_{b}(0) | 10 \rangle \]

\[ \langle 01 \phi_{b}(0) | 10 \rangle \neq 0 \Rightarrow C_{a}^{A} \neq 0 \]

\[ \Rightarrow \text{states with mass } \mu = 0. \]