5.2. Nonrelativistic QED/QCD

Let us now consider the effective theory relevant for the description of bound states of two heavy particles, e.g. positronium (e^+e^-), muonium (μ^+e^-), bottomonium ($b\bar{b}$), charmonium ($c\bar{c}$).

The effective theories are called NRQED and NRQCD respectively and are closely related to HQET, except that we now deal with both a particle, described by a two-component spinor ψ, and an anti-particle which we denote by χ. The effective Lagrangian has the form

$$\mathcal{L}_{NR} = \mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{mixed} + \mathcal{L}_{light}.$$

$\mathcal{L}_{mixed}$ contains operators involving both χ and ψ

$\mathcal{L}_{light}$ is the QCD Lagrangian for the light quarks plus higher dim. operators.

The Lagrangians for the ψ field is nothing but the HQET Lagrangian evaluated for $v^\mu = (1, \vec{0})$ since it is natural to work in the rest frame of the bound state. So

$$\mathcal{L}_\psi = \psi^+ \left(iD_t + \frac{\vec{D}^2}{2m_\psi} \right) \psi + \frac{1}{8m_\psi^3} \psi^+ \vec{D}^4 \psi$$

$$+ \frac{gc_1}{2m_\psi} \psi^+ \vec{\sigma} \cdot \vec{B} \psi + \frac{c_2 g}{8m_\psi^2} \psi^+ (\vec{D} \cdot \vec{E} - \vec{E} \cdot \vec{D}) \psi$$

$$+ \frac{gc_3}{8m_\psi^2} \psi^+ (i\vec{D} \times \vec{E} - \vec{E} \times i\vec{D}) \cdot \vec{\sigma} \psi$$

[$g \equiv g_s$, or $g = -e$ is the gauge coupling.]

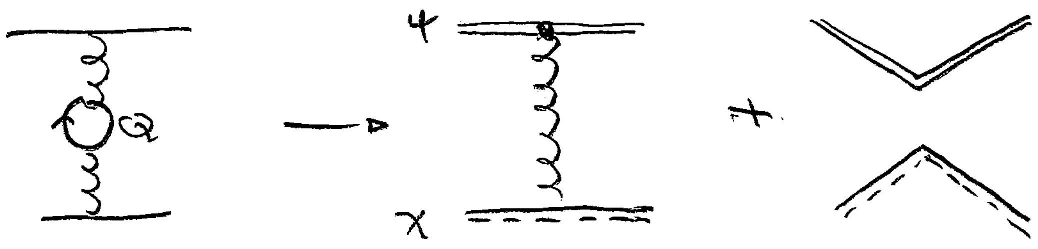
We have included $1/m_Q^2$ and $1/m_Q^3$ suppressed terms because the power counting is different than in HQET.

$$\mathcal{L}_\chi = \mathcal{L}_\psi \Big|_{\substack{\psi \to \chi \\ A^\mu \to -A^\mu}} = \text{"charge conj. of } \mathcal{L}_\psi \text{"}$$

The lowest dimensional operators in $\mathcal{L}_{mixed}$ are four-quark operators. For example, it contains

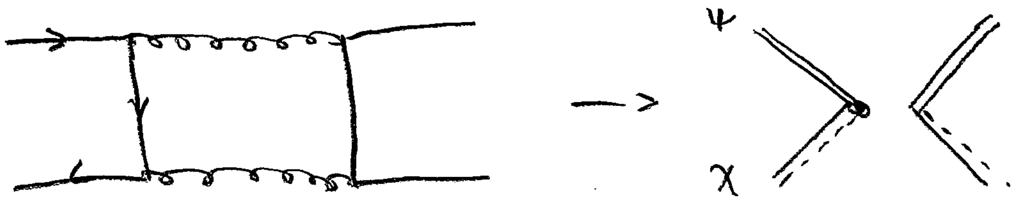
$$\mathcal{L}_{mixed} = + \frac{c_4}{m^2} \psi^+ \psi \chi^+ \chi + \frac{c_5}{m^2} \psi^+ \vec{\sigma} \sigma_2 \chi^* \times \chi^+ \sigma_2 \vec{\sigma} \psi$$

The first operator arises, when high energy contributions to the scattring of χ and ψ are integrated out, e.g.



$(\psi^\dagger\psi)(\chi^\dagger\chi)$

The second one arises in annihilation diagrams such as



$(\chi^T\sigma_2\vec{\sigma}\psi)\,(\psi^\dagger\vec{\sigma}\sigma_2\chi^*)$

These diagrams only exist if χ is the anti-particle of ψ. They have an imaginary part, which describes the decay $\psi\chi \to \gamma\gamma$ (or gg). $\Rightarrow$ c_5 is imaginary. The effective H is not hermitian and the theory is not unitary!

However, there is a good physical reason for this violation of unitary: bound states, such as e^+e^- decay over time. The imaginary part of H encodes the decay rate. The probability for finding the e^- in e^+e^- is not 1 for all times, because it will annihilate sooner or later.

This is the first complication compared to HQET. The second one is that the static Lagrangian

$$\mathcal{L} = \psi^+ i\partial_t \psi$$

cannot serve as a starting point in NR theories. There are formal arguments to show this, but the simple physical reason is that the e^+e^- in the bound state are not static. They are close to their mass-shell $E = \frac{p^2}{2m_+} + \dots$ and we thus should count $D_t \sim \frac{D^2}{2m} \sim \frac{m\vec{v}^2}{2}$ as of the same order. Instead of powers of $\frac{1}{m_Q}$, we should count powers of $v = |\vec{v}|$.

A third complication is that that multiple photon/gluon exchanges between ψ and χ are unsuppressed. More precisely, the exchange of Coulomb gluons needs to be taken into account to all orders.

In Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, the gauge Lagrangian reads

$$-\frac{1}{4} G^{\mu\nu} G_{\mu\nu} = \frac{1}{2} G^{0i} G_{0i} - \frac{1}{4} G^{ij} G_{ij}$$

$$= \frac{1}{2}\left[(\partial_i A_0)^2 + (\partial_0 A_i)^2 - (\partial_i A_j)^2 + \text{"non-abelian"} \right]$$

The field A_0 has no time derivatives and is thus not propagating. Since the action is quadratic, one can integrate out A^0. It's effect is then described by a potential, which is just the Fourier transform of its propagator

$$V(\vec{x}-\vec{y}) = g_s^2 \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}(\vec{x}-\vec{y})} \frac{1}{\vec{k}^2} = \frac{g_s^2}{4\pi |\vec{x}-\vec{y}|}.$$

The leading power effective Lagrangian for non-relativistic particle - antiparticle pair is then

$$L_{NR} = \int d^3x \; \psi^\dagger \left[i\partial_t + \frac{\vec{\nabla}^2}{2m} \right] \psi$$

$$- \int d^3x_1 \int d^3x_2 \; \psi^\dagger(x_1)\, t^a\, \psi(x_1)\, \chi^\dagger(x_2)\, t^a\, \chi(x_2)\, V(\vec{x}_1 - \vec{x}_2)$$

Accounting for $V(\vec{x})$ to all orders amounts to solving the Schrödinger equation. The remaining terms are treated as perturbations.

Unfortunately, the fact that the problem involves the three scales

	m (hard)	mv (soft)	mv^2 (ultrasoft)
e^+e^- value	0,5 MeV	3,7 keV	25 eV

(where we have used that for positronium $v \sim \alpha$)

makes it difficult to organise the computations.

In particular, in dim. reg. the NR integrals get contributions from the hard region, since the scale m_Q appears in the integrand. Initially people used to perform the computations with a hard cut-off, which avoids this problem but makes computations extremely cumbersome.

Using the "threshold expansion" (Beneke and Smirnov '98) which is also called the "strategy of regions" it became possible to eliminate the unwanted hard corrections in dim. reg. and to separate the soft and ultrasoft corrections.

An effective theory approach which implements this separation on the level of the Lagrangian is "velocity NRQCD" or vNRQCD, first proposed by Luke, Manohar and Rothstein '99.

Earlier Pineda and Soto '97 had proposed "potential NRQCD" (pNRQCD). Their idea was to integrate out the soft scale mv and to construct an effective theory containing only ultrasoft degrees of freedom:

$$\mathcal{L}_{QCD} \quad (\mu > m) \qquad h + s + us$$

$$\downarrow$$

$$\mathcal{L}_{NRQCD} \ (m > \mu > mv) \qquad s + us$$

$$\downarrow$$

$$\mathcal{L}_{pNRQCD} \ (mv > \mu > mv^2) \quad us$$

The fields in pNRQCD are not quarks and anti-quarks but color singlet and color octed $\bar{Q}Q$-pairs:

$$S \equiv S(r) \sim \chi^+(-\tfrac{r}{2})\psi(\tfrac{r}{2}); \qquad O^a \sim \chi^+ t^a \psi$$

(singlet) (octet)

which interact through potentials $V(r)$ and ultrasoft gluons.