5.1.2. Applications of HQET

Let us now derive some properties of heavy mesons from HQET. The heavy quark inside heavy-light hadrons, such as B- or D-mesons is almost static. Neglecting the $1/m_Q$ corrections to $\mathcal{L}_{eff}$, we have

$$\mathcal{L}_{eff}^{(0)} = \bar{h}_v \, iv.D \, h_v + \text{"QCD for light quarks"}$$

This leading Lagrangian now exhibits heavy quark symmetry: it is independent of m_Q and the quark gluon interaction is spin-independent. Eigenstates of $\mathcal{H}_{eff}$ will thus come in degenerate doublets which differ by the spin state of the heavy quark. Since the total angular momentum

$$J = S_e + S_Q$$

is conserved, also the spin of the light degrees of freedom is conserved.

It is conventional to normalize the states in HQET

as $$\langle H(r) | H(r') \rangle = \underset{\sim\sim\sim}{2v^0}\, (2\pi)^3\, \delta^{(3)}(\vec{r} - \vec{r}\,') \qquad (*)$$

i.e to use $2v^0$ instead of the factor $2E_k$ which is used in relativistic field theory.

To compute meson masses one computes the expectation value of the Hamiltonian (density) $\mathcal{H}$ for a meson at rest : $v^\mu = (1, 0)$, $r = 0$.

$$\bar{\Lambda}_\ell = \frac{1}{2} \langle H(0) | \mathcal{H}^{(0)} | H(0) \rangle$$

from normalization $(*)$

where $\mathcal{H}^{(0)}$ is the leading order Hamiltonian obtained from $\mathcal{L}^{(0)}_{eff} = \bar{h}_v\, i v D\, h_v +$ "light d.o.f". The quantity $\bar{\Lambda}_\ell$ is not the mass, since we have split $p = m_Q v + r$,

$$M_H = m_Q + \bar{\Lambda}_\ell + O\left(\frac{1}{m_Q}\right)$$

The index ℓ reminds us that $\bar{\Lambda}_\ell$ depends on the configuration of the light degrees of freedom but is independent of the spin of the heavy quark.

To obtain the $1/m_Q$ corrections to the mass we need to take the matrix element of

$$\mathcal{H}^{(1)} = -\mathcal{L}^{(1)} = \bar{h}_v \frac{D_\perp^2}{2m_Q} h_v + C_{mag}(\mu) \frac{g_s}{4m_Q} \bar{h}_v \sigma^{\mu\nu} G_{\mu\nu} h_v$$

$C_{mag}(\mu) = 1 + O(\alpha_s)$ is the Wilson coeff. of the chromomagnetic operator. One can show that Lorentz invariance implies that the kinetic operator has unit Wilson coefficient ($E = \frac{p^2}{2m} + \dots$ must hold, independent of quantum effects.) The matrix elements of the two operators are

$$\lambda_1^{\ell} = -\frac{1}{2} \langle H(0) | \bar{h}_v D_\perp^2 h_v | H(0) \rangle$$

$$16 (\vec{S}_Q \cdot \vec{S}_\ell) \lambda_2^{\ell} = C_{mag} \langle H(0) | g_s \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v | H(0) \rangle$$

We have seen in the previous section that

$$-\frac{1}{4} \bar{h} \sigma_{\mu\nu} G^{\mu\nu} h = + \bar{h} \frac{\vec{\Sigma}}{2} h \vec{B}_c = \vec{S}_Q \cdot \vec{B}_c$$

where $\vec{B}_c$ is the chromomagnetic field. Since the matrix element is a scalar the matrix element of $\vec{B}_c$ must then be proportional to the spin of the light d.o.f.

The matrix element of the chromomagnetic operator must thus be proportional to the expectation value of $\vec{S}_Q \cdot \vec{S}_e$. Let's rewrite

$$2\vec{S}_Q \cdot \vec{S}_e = \vec{J}^2 - \vec{S}_Q^2 - \vec{S}_e^2$$

$$d_J = 4\langle \vec{S}_Q \cdot \vec{S}_e \rangle = 2\left[j(j+1) - s_Q(s_Q+1) - s_e(s_e+1)\right]$$
$$= 2\left[j(j+1) - \frac{3}{4} - s_e(s_e+1)\right]$$

Some examples

$$M_B = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} - \frac{3\lambda_2}{2m_b}$$

$\Big[$ B has spin $j=0$, $s_e = \frac{1}{2}$ $\Rightarrow d_J = 2\left[-2\times\frac{3}{4}\right] = -3$ $\Big]$

$$M_{B^*} = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} + \frac{\lambda_2}{2m_b}$$

$\Big[$ B^* has spin $j=1$: $d_J = 2\left[2 - \frac{3}{2}\right] = 1$

$$M_{\Lambda_b} = m_b + \bar{\Lambda}_\Lambda - \frac{\lambda_1^\Lambda}{2m_b} + 0$$

$\Big[$ Λ_b has spin $j = \frac{1}{2}$, $s_e = 0$: $d_J = 2\left[\frac{3}{4} - \frac{3}{4}\right] = 0$

Numerically this works well:

$$(M_{B^*} - M_B) / (M_{D^*} - M_D) = 0,32 \approx \frac{m_c}{m_b} \approx \frac{M_D}{M_B} = 0,35$$

$$M_{H^*}^2 - M_H^2 = 4\lambda_2 + O\left(\frac{1}{m_b}\right)$$

$$M_{B^*}^2 - M_B^2 = 0,49\ \mathrm{GeV}^2$$

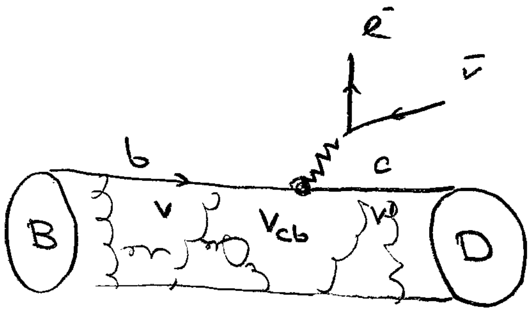
$$M_{D^*}^2 - M_D^2 = 0,55\ \mathrm{GeV}^2$$

$$M_{\Lambda_b} - \frac{1}{4}\left(3M_{B^*} + M_B\right) = 306\ \mathrm{MeV}$$

$$M_{\Lambda_c} - \underbrace{\frac{1}{4}\left(3M_{D^*} + M_D\right)}_{\substack{\text{spin averaged,} \\ \text{indep. of } \lambda_2.}} = 314\ \mathrm{MeV}$$

See figures for a comparison of the B and D meson spectrum.

As a second application we will compute the semileptonic decay $\bar{B} \to D \, \ell^- \bar{\nu}$



Clever use of heavy quark symmetry will allow us to obtain the rate at one particular kinematical point, the "zero recoil point" where $v^\mu \equiv \frac{p_B^\mu}{M_B} = v'^\mu \equiv \frac{p_D^\mu}{M_D}$. At this point the momentum transfer to the lepton pair is maximal.

Let us first consider the vector form factor of a B-meson:

$$\langle \bar{B}(p') \,|\, \bar{b} \gamma^\mu b \,|\, \bar{B}(p) \rangle = F(q^2)(p+p')^\mu$$

Because of current conservation $F(0) = 1$.

At leading power in $1/m_Q$, the vector current in HQET takes the form

$$\langle \bar{B}(v') | \bar{b}_{v'} \gamma^\mu b_v | B(v) \rangle = \xi(v \cdot v') \, (v+v')^\mu$$

HQET field

Isgur-Wise function

comparison with the QCD result gives

$$\lim_{m_b \to \infty} F(q^2) = \xi(v \cdot v')$$

$$q^2 = (p_B' - p_B)^2 = -2m_B^2 \, (v \cdot v' - 1) \leq 0$$

and vector current conservation implies

$$F(0) = 1 \iff \xi(1) = 1$$

Note that the states in HQET are normalized differently than in QCD:

$$\langle \bar{B}(p') | B(p) \rangle = m_B \langle \bar{B}(v') | \bar{B}(v) \rangle$$

So far, so trivial ...

Things become interesting when we use heavy quark symmetry. In the limit $m_Q \to \infty$, the $B \to B$ and $B \to D$ vector form factors are equal

$$\langle D(v') | \bar{c}_{v'} \gamma^\mu b_v | \bar{B}(v) \rangle = \xi(v \cdot v') (v + v')^\mu$$

This is interesting, since the most general parameterization of this matrix element involves <u>two</u> form factors $f_\pm$

$$\langle D(p') | \bar{c} \gamma^\mu b | B(p) \rangle = f_+(q^2)(p+p')^\mu + f_-(q^2)(p-p')^\mu$$

normalization of states

$$= \sqrt{m_B m_D}\, \xi(v \cdot v')(v+v')^\mu + O\left(\frac{1}{m_Q}\right)$$

So heavy-quark symmetry <u>relates</u> the two form factors and <u>predicts their normalization</u> for $v = v'$:

$$f_\pm(q^2) = \frac{m_B \pm m_D}{2\sqrt{m_B m_D}}\, \xi(v \cdot v')$$

$$q^2 = m_B^2 + m_D^2 - 2 m_B m_D\, v \cdot v' \geq 0; \quad q^2_{max} = (m_B - m_D)^2$$

$$\Rightarrow f_\pm(q^2_{max}) = \frac{m_B \pm m_D}{2\sqrt{m_B m_D}} \times 1 .$$

One can furthermore make use of the heavy-quark spin symmetry to also express the four $B \to D^*$ vector form factors in terms of $\xi(v \cdot v')$.

These form factors relations get corrections of order

$$\underbrace{\Lambda_{QCD}/m_Q}_{\text{nonperturbative}} \qquad \text{and} \qquad \underbrace{\alpha_s(m_Q)}_{\text{calculable matching corrections}}$$

The semileptonic $\bar{B} \to D^{(*)} \ell \nu$ rate is obtained from

$$\mathcal{H}_{SL} = \frac{G_F}{\sqrt{2}} V_{cb}\, \bar{c}\gamma_\mu(1-\gamma_5)b\ \bar{e}\gamma^\mu(1-\gamma_5)\nu_e$$

$$\mathcal{A} = \frac{G_F}{\sqrt{2}} V_{cb} \overbrace{\langle D^{(*)}(v') \,|\, \bar{c}\gamma^\mu(1-\gamma_5)b \,|\, B(v)\rangle}^{\text{Hadronic matrix elements } \propto\ \xi(v \cdot v')}$$

$$* \ \bar{u}_e(p_e)\gamma^\mu(1-\gamma_5)v_\nu(p_\nu)$$

To get the rate, one has to square $\mathcal{A}$, sum over spins, and integrate over phase-space.

In terms of the variable

$$w = v \cdot v' = \frac{m_B^2 + m_D^2 - q^2}{2 m_B m_D} \in \left[1, \frac{m_B^2 + m_D^2}{2 m_B m_D}\right]$$

one obtains for $B \to D \ell \nu$

$$\frac{d\Gamma}{dw} = \frac{G_F^2}{48\pi^3} |V_{cb}|^2 (m_B + m_D)^2 m_D^3 (w^2 - 1)^{3/2} \xi^2(w)$$

with $\xi(1) = 1$. A similar expression holds for $B \to D^* \ell \nu$. To determine $|V_{cb}|$, one measures the decay rate, and extracts $|V_{cb}|$ from an extrapolation to $w = 1$. The $B \to D^* \ell \nu$ is particularly suitable, since it does not receive first order $1/m_Q$ corrections at $w = 1$ (Luke's theorem). Furtermore, the phase space suppression is $(w^2-1)^{1/2}$ instead of $(w^2-1)^{3/2}$ in this case.

The $1/m_b$ corrections can be extracted from lattice simulations, after which one obtains one of the most precise determinations of $|V_{cb}|$.