5.1.1. Connection to Quantum Mechanics

Let us go into the rest frame of the heavy quark $v^\mu = (1, \vec{0})$. The projection operator is then

$$P_+ = \frac{1}{2}(\mathbb{1} + \gamma^0) = \frac{1}{2}\left(\mathbb{1} + \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\right) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

i.e. P_+ projects out the upper two components of the Dirac field. Also, let us consider QED instead of QCD: $\psi_Q \to \psi_e$; $iD_\mu \to i\partial_\mu - eA_\mu$.

The magnetic operator becomes

$$-\frac{e}{4m_e}\bar{h}_v \sigma^{\mu\nu} F^{\perp}_{\mu\nu} h_v = -\frac{ie}{8m_e}\bar{h}\overbrace{(\sigma^i\sigma^j - \sigma^j\sigma^i)}^{2i\varepsilon^{ijk}\sigma^k}\overbrace{(\partial_i A_j - \partial_j A_i)}^{2\partial_i A_j} h$$

$$= +\frac{e}{2m_e}\bar{h}\,\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A})\,h = +\frac{e}{2m_e}\bar{h}_v\vec{\sigma}\cdot\vec{B}\,h_v .$$

The effective Lagrangian for a slow electron is

$$\mathcal{L} = \bar{\chi}\, iD_t\, \chi - \bar{\chi}\frac{(i\vec{D})^2}{2m_e}\chi + \underbrace{\frac{e}{2m_e}\bar{\chi}\,\vec{\sigma}\cdot\vec{B}\,\chi}_{2\vec{\mu}\cdot\vec{B}} .$$

where $h_v = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$.

$\uparrow$ magnetic moment

The EOM associated with this Lagrangian is the Schrödinger equation for an e^- interacting with a photon field.

The free propagator associated with $\mathcal{L}$

$$\Delta_x = \frac{1}{E - \frac{\vec{p}^2}{2m} + i\varepsilon}$$

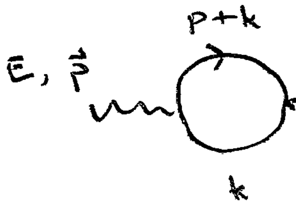
has only a <u>single</u> pole, in contrast to a relativistic propagator:

$$\frac{1}{p^2 - m^2 + i\varepsilon} = \frac{1}{2\omega}\Bigg[\underbrace{\frac{1}{p^0 - \omega + i\varepsilon}}_{\text{particle}} - \underbrace{\frac{1}{p^0 + \omega - i\varepsilon}}_{\text{anti-particle}}\Bigg]$$

$$= \frac{1}{2m}\,\frac{1}{E - \frac{\vec{p}^2}{2m} + i\varepsilon} + \cdots$$

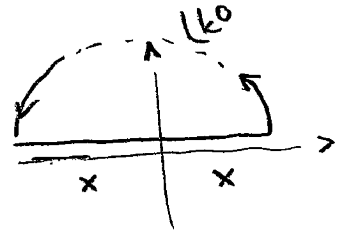
where $\omega = m + \frac{p^2}{2m} + \cdots$, $p^0 = m + E$.

This has important consequences: because the theory no longer contains anti-particles, closed fermion loops vanish:



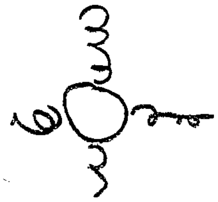
$$\propto \int d^d k \; \frac{1}{(E+k^0) - \frac{(\vec{p}+\vec{k})^2}{2m} + i\varepsilon} \; \frac{1}{k^0 - \frac{\vec{k}^2}{2m} + i\varepsilon}$$

$= 0$ because we can close the k^0 integration contour



without encountering a pole since $\mathrm{Im}\, k^0 < 0$ for all poles.

$\Rightarrow$ All fermion loops vanish in HQET. The effect of virtual antiparticles can be absorbed into the Wilson coefficients of the operators in $\mathcal{L}_{eff}$, e.g.



is represented by the Euler-Heisenberg terms in $\mathcal{L}_{eff}$.

Despite this, our theory is not simply QM, since the electromagnetic field is a fully relativistic quantum field. To obtain QM, we need to treat the e.m. field as classical. Let's therefore assume that $A^\mu = (\phi(\vec{x}), 0)$ is a fixed classical potential (e.g. the Coulomb field of a proton).

Now the field operator fulfills

$$i\partial_t \hat{\chi} = [\hat{\chi}, \hat{H}] = \left(-\frac{\vec{\nabla}^2}{2m} + e\phi(\vec{x})\right)\hat{\chi}$$

The solutions of the time-independent Schrödinger equation

$$\left(-\frac{\vec{\nabla}^2}{2m} + e\phi(\vec{x})\right)\varphi_n(x) = E_n \varphi(x)$$

form a complete set of functions which can be used to expand

$$\hat{\chi}(t,\vec{x}) = \sum_i \hat{a}_i e^{-iE_i t} \varphi_i(x)$$

The operator $\hat{a}_i$ annihilates the state with associated wave function $\varphi_i(x)$.

Now the system is indeed quantum mechanical:

the one-particle states are $|i\rangle = a_i^\dagger |0\rangle$

and they have associated wave functions

$$\langle 0 | \hat{\chi}(t, \vec{x}) | i \rangle = e^{-iE_i t} \varphi_i(x)$$

which fulfill the Schrödinger equation.

Let us briefly recapitulate.

* The EFT for a nonrelativistic particle has a Lagrangian which has the Schrödinger eqn. as EOM.

* There are no antiparticles in the EFT, their effect can be absorbed into the Wilson coefficients, since they are highly virtual.

* Treating the photon as a classical background field we recover quantum mechanics.