5.1.1. Connection to Quantum Mechanics

Let us go into the rest frame of the heavy quark $v^k = (1, \vec{0})$. The projection operator is then

$$P_+ = \frac{1}{2} (1 + \gamma_0) = \frac{1}{2} (1 + (\gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 \gamma^6 \gamma^7 \gamma^8)), \nonumber$$

i.e. $P_+$ projects out the upper two components of the Dirac field. Also, let us consider QED instead of QCD: $\bar{\psi} \psi \rightarrow \bar{\psi} \psi \quad iD_\mu \rightarrow iD_\mu - eA_\mu$.

The magnetic operator becomes

$$\frac{e}{4m_e} \bar{u}_v \sigma^{\mu \nu} F_{\mu \nu} u_v = -\frac{ie}{8m_e} \bar{u}_v (\sigma^i \sigma^j - \sigma^j \sigma^i) (\partial_i A_j - \partial_j A_i) u_v \nonumber$$

$$= +\frac{e}{2m_e} \bar{u}_v \vec{\sigma} \cdot (\vec{\nabla} \times \vec{A}) u_v = +\frac{e}{2m_e} \bar{u}_v \vec{\sigma} \cdot \vec{B} u_v. \nonumber$$

The effective lagrangian for a slow electron is

$$\mathcal{L} = \bar{\chi} iD_\mu X - \bar{\chi} \left( \frac{i\vec{\sigma}}{2m_e} \right)^2 X + \frac{e}{2m_e} \bar{\chi} \vec{\sigma} \cdot \vec{B} X. \nonumber$$

where $h_v = \left( \frac{X}{0} \right)$.
The EOM associated with this Lagrangian is the Schrödinger equation for an $e^{-}$ interacting with a photon field.

The free propagator associated with $\mathcal{L}$

$$\Delta_x = \frac{1}{E - \frac{p^2}{2m} + i\varepsilon}$$

has only a single pole, in contrast to a relativistic propagator:

$$\frac{1}{p^2 - m^2 + i\varepsilon} = \frac{1}{2m} \left[ \frac{1}{p^0 - \omega + i\varepsilon} - \frac{1}{p^0 + \omega - i\varepsilon} \right]$$

$$= \frac{1}{2m} \frac{1}{E - \frac{p^2}{2m} + i\varepsilon} + \ldots$$

where $\omega = m + \frac{p^2}{2m} + \ldots$, $p^0 = m + E$. 
This has important consequences: because the theory no longer contains anti-particles, closed fermion loops vanish:

\[
E, p, m \propto \int \frac{1}{(E + k^0) - \left( \frac{p + E}{2m} \right) + i\varepsilon} \frac{1}{k^0 - \frac{E^2}{2m} + i\varepsilon}
\]

= 0 because we can close the \( k^0 \) integration contour without encountering a pole since \( \text{Im} k^0 < 0 \) for all poles.

=> All fermion loops vanish in HQET. The effect of virtual anti-particles can be absorbed into the Wilson coefficients of the operators in \( \Delta f \), e.g.

\( c_{\omega} \) is represented by the Euler-Heisenberg terms in \( \Delta f \).
Despite this, our theory is not simply QM, since the electromagnetic field is a fully relativistic quantum field. To obtain QM, we need to treat the r.m. field as classical. Let's therefore assume that $A^μ = (\phi(\vec{x}), 0)$ is a fixed classical potential (e.g., the Coulomb field of a proton).

Now the field operator fulfills

$$i \hbar \dot{\hat{X}} = [\hat{X}, \hat{H}] = \left( -\frac{\hbar^2}{2m} + e\phi(\vec{x}) \right) \hat{X}$$

The solutions of the time-independent Schrödinger equation

$$\left( -\frac{\hbar^2}{2m} + e\phi(\vec{x}) \right) \psi_n(x) = E_n \psi_n(x)$$

form a complete set of functions which can be used to expand

$$\hat{X}(t, \vec{x}) = \sum_i \hat{\alpha}_i e^{-iE_i t} \psi_i(x)$$

The operator $\hat{\alpha}_i$ annihilates the state with associated wave function $\psi_i(x)$. 
Now the system is indeed quantum mechanical: the one-particle states are $|i\rangle = q_i^+ |0\rangle$
and they have associated wave functions

$$\langle 0 | \hat{X}(t, \vec{x}) | i \rangle = e^{-iE_i t} \varphi_i(x)$$

which fulfill the Schrödinger equation.

Let us briefly recapitulate.

* The EFT for a nonrelativistic particle
  has a Lagrangian which has the Schrödinger
equation as EOM.

* There are no antiparticles in the EFT; their
  effect can be absorbed into the Wilson
  coefficients, since they are highly virtual.

* Treating the photon as a classical background
  field we recover quantum mechanics.