5. Nonrelativistic effective theories

We have considered several effective theories which are obtained by integrating out heavy particles. However, in many cases* heavy particles are present even at very low energy. The reason are conservation laws for particle numbers such as lepton number conservation $L = L_{e^-} - L_{e^+}$ and baryon (or quark) number conservation. If we neglect the weak interaction then each lepton and quark flavor is separately conserved.

The proper framework to describe heavy particles at low momentum are nonrelativistic effective theories. Examples of systems which can be studied with such techniques are atoms, mesons with heavy (i.e. bottom or charm) quarks, and protons interacting with slow pions, etc.

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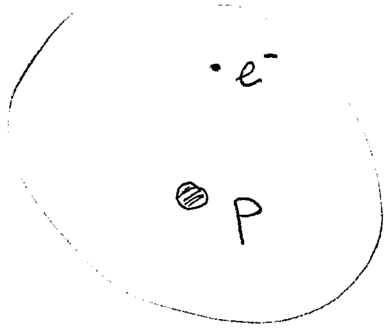
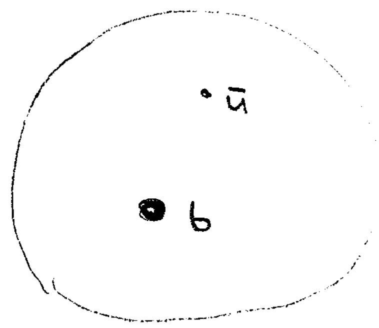
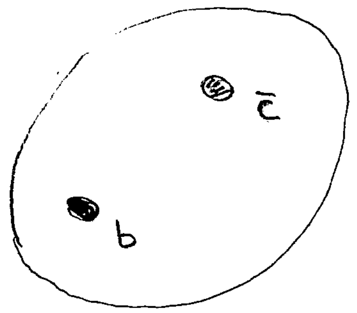
* such as the real world ...



Hydrogen

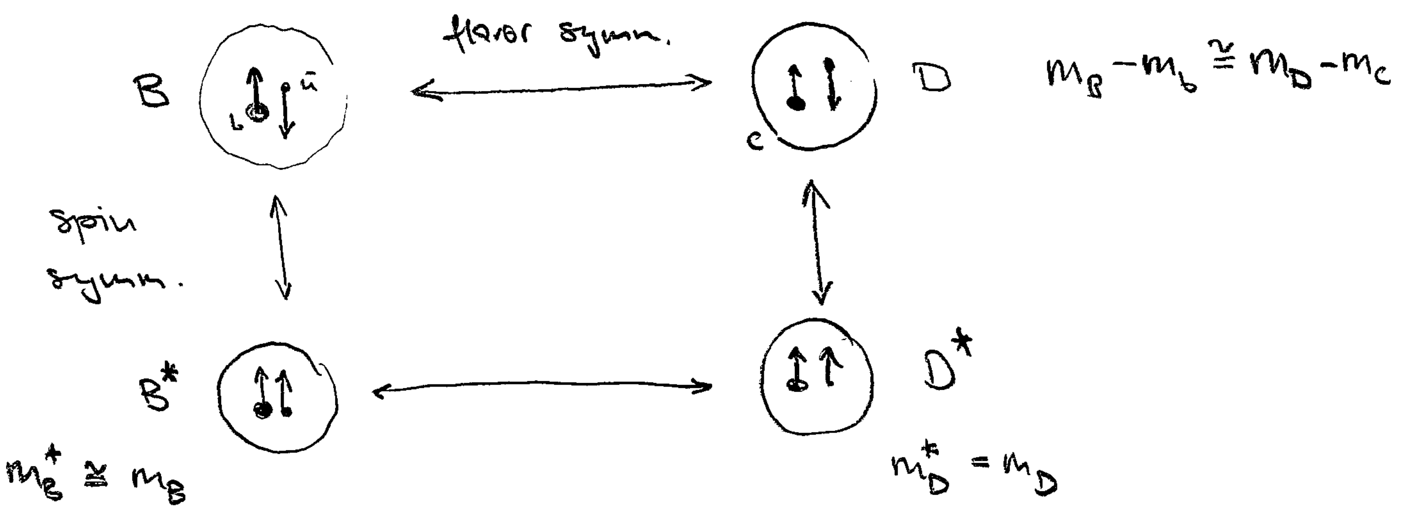


B^- - meson



B_c - meson

A heavy B-meson has similarities to a hydrogen atom, but an important difference is that the light degrees of freedom inside the B-meson are still highly relativistic and strongly interacting. Nevertheless some properties of hydrogen carry over: the energy of the B-meson is to good accuracy independent of the b-quark spin. Also, the energy spectrum of B-mesons is independent of the heavy quark mass to good approximation:



Heavy Quark Effective Theory (HQET) will allow us to derive such relations in the limit $m_Q \to \infty$ and to systematically analyze the $1/m_Q$ corrections. For systems such as hydrogen or a B_c meson also the lighter fermion (e^- or $\bar{c}$ respectively) can be treated nonrelativistically. The EFT for this case is Non-Relativistic QED/QCD (NRQED/NRQCD). HQET and NRQCD have the same Lagrangian but different power counting.

5.1. Heavy-Quark Effective Theory (HQET)

Interactions of the heavy quark Q with the light constituents of a heavy-to-light meson will change its momentum by amounts of order $\Lambda_{QCD} \sim 1\text{GeV}$, but it's velocity is barely changed $\delta v_Q^\mu = \frac{\Delta p_Q^\mu}{m_Q} \ll 1$. To analyze such systems, we introduce a refrence vector v^μ, $v^2 = 1$, in the direction of the heavy quark and split

$$p_Q^\mu = m_Q v^\mu + r^\mu$$

so that the residual momentum r^μ is $O(\Lambda_{QCD})$.

A popular choice for v^μ is the meson velocity $v^\mu = \frac{P_M^\mu}{m_M}$. The effective field theory is an expansion in the residual momentum r^μ over the heavy quark mass m_Q.

On the level of the quark field the decomposition of the momentum is achieved by splitting off the large phase $e^{-im_Q v\cdot x}$ from the field:

$$\psi_Q(x) = e^{-im_Q v\cdot x}\left\{ h_v(x) + H_v(x) \right\}$$

where

$$h_v(x) = e^{imvx}\, P_+\, \psi(x) \quad ; \quad P_+ = \frac{1+\not{v}}{2}$$

$$H_v(x) = e^{imvx}\, P_-\, \psi(x) \quad ; \quad P_- = \frac{1-\not{v}}{2}$$

The projection operators P_+ and P_- split the field into the large ("upper") components $h_v(x)$ and the small ("lower") components $H_v(x)$. They obey

$$\not{v}\, h_v(x) = h_v(x)$$

$$\not{v}\, H_v(x) = -H_v(x).$$

Let's insert this decomposition into the Dirac Lagrangian: $\mathcal{L}_Q = \bar{\psi}(i\not{D} - m_Q)\psi$

$$= \bar{h}_v\, i\not{D}\, h_v + \bar{H}_v(i\not{D} - 2m_Q)H_v + \bar{H}_v i\not{D} h_v + \bar{h}_v i\not{D} H_v$$

This simplifies further to

$$\mathcal{L}_Q = \bar h_v \, i v\cdot D \, h_v + \bar H_v(-i v\cdot D - 2m_Q) H_v$$

$$+ \bar H_v i \not{D}_\perp h_v + \bar h_v i \not{D}_\perp H_v .$$

where $D_\perp^\mu = D^\mu - i v\cdot D v^\mu$ are the components perpendicular to v^μ. Here we used that

$$\bar h_v \gamma^\mu h_v = \bar h_v \gamma^\mu \not{v} h_v = -\bar h_v \gamma^\mu h_v + 2 v^\mu \bar h_v h_v$$

$$\Rightarrow \bar h_v \gamma^\mu h_v = v^\mu \bar h_v h_v$$

Similarly: $\bar H_v \gamma^\mu H_v = - v^\mu \bar H_v H$.

and $\bar H_v \not{v} h_v = \bar h_v \not{v} H_v = 0$.

The equation of motion for H_v is

$$(-i v\cdot D - 2m_Q) H_v + i \not{D}_\perp h_v = 0$$

$$\Rightarrow H_v = \frac{1}{2m_Q} \sum_{n=0}^{\infty} \left(- \frac{i v\cdot D}{2m_Q} \right)^n i \not{D}_\perp h_v$$

So H_v is suppressed with respect to h_v by a factor $r_\perp / 2m_Q$ and can be integrated out. Since the action is quadratic this can be done exactly.

At the classical level, the result is obtained by plugging the solution of the EOM for H_v back into $\mathcal{L}_Q$. One obtains

$$\mathcal{L}_Q = \bar{h}_v i v\cdot D h_v + \frac{1}{2m_Q}\bar{h}_v \underbrace{i\not{D}_\perp i\not{D}_\perp h_v}_{\text{power corrections}} + O\left(\frac{1}{m_Q^2}\right)$$

Rewrite:

$$\begin{aligned} i\not{D}_\perp i\not{D}_\perp &= iD^\perp_\mu iD^\perp_\nu \left(\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \tfrac{1}{2}[\gamma^\mu,\gamma^\nu] \right) \\ &= iD^\perp_\mu iD^\perp_\nu \left(g_{\mu\nu} - i\sigma^{\mu\nu} \right) \\ &= (iD_\perp)^2 + \frac{i}{2}[D^\perp_\mu, D^\perp_\nu]\sigma^{\mu\nu} \\ &= (iD_\perp)^2 + \frac{g_s}{2}\sigma^{\mu\nu}G_{\mu\nu} \end{aligned}$$

$$\Rightarrow \mathcal{L}_Q = \bar{h}_v i v\cdot D h_v + \frac{1}{2m_Q}\bar{h}(iD_\perp)^2 h + \frac{g_s}{4m_Q}\bar{h}_v \sigma_{\mu\nu}G^{\mu\nu} h$$

In the rest frame, $v^\mu = (1,0)$

$$\mathcal{L}_Q = \bar{h}_v iD_t h_v - \underset{\underset{\text{kinetic energy}}{\uparrow}}{\frac{1}{2m}\bar{h}_v(i\vec{D})^2 h_v} - \underset{\underset{\text{chromomagnetic moment.}}{\uparrow}}{\frac{g_s}{2m_Q}\bar{h}\,\vec{\sigma}\cdot\vec{B}_c\, h}$$