4.4.4. Applications

We now consider two applications: $\pi\pi$-scattering and the one-loop corrections to the relation $M_\pi^2 = (m_u + m_d) B_0$. In each case, the starting point is to expand $\mathcal{L}_{eff}$ in powers of the π-field and to derive the Feynman rules for the corresponding vertices. As before we expand

$$U(x) = \exp\left[\frac{i}{F}\pi\right] = 1 + \frac{i}{F}\pi - \frac{1}{2F^2}\pi^2 + \dots$$

where $\pi = \pi^a \tau^a$.

We need $\mathcal{L}^{(2)}$ up to terms with four π-fields:

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \mathrm{tr}\left[\partial_\mu U \partial^\mu U\right] + \frac{F^2 B_0}{2} \mathrm{tr}\left[m(U + U^+)\right]$$

$$= \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{1}{48F^2} \mathrm{tr}\left\{[\partial_\mu \pi, \pi][\partial^\mu \pi, \pi]\right\}$$

$$+ \frac{F^2 B_0}{2} \mathrm{tr}\left[m\left(1 - \frac{\pi^2}{F^2} + \frac{\pi^4}{12F^4}\right)\right]$$

Evaluating the traces, one finds

$$\mathcal{L}^{(2)} = \frac{1}{2}\partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi} - \frac{M^2}{2}\vec{\pi}^2 + \frac{1}{6F^2}\left[(\vec{\pi}\,\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi})^2\right]$$

$$+ \frac{M^2}{24F^2}(\vec{\pi}^2)^2$$

where $M = B_0(m_u + m_d)$. To compute the corrections to the π-mass we also need to expand $\mathcal{L}^{(4)}$ to second order in the π-field. The only relevant terms are (the full $\mathcal{L}^{(4)}$ is given in Gasser and Leutwyler '84)

$$\mathcal{L}^{(4)} = \frac{\ell_3 B_0^2}{4}\left(\mathrm{tr}[u^+ m + m^+ u]\right)^2$$

$$- \frac{\ell_7 B_0^2}{4}\left(\mathrm{tr}[u^+ m - m^+ u]\right)^2 + \dots$$

$$= \frac{\ell_3 B_0^2}{4}\left(\mathrm{tr}[m](2 - \frac{\vec{\pi}^2}{F^2})\right)^2$$

$$- \frac{\ell_7 B_0^2}{4}\left(\mathrm{tr}\left[m \frac{2i\vec{\sigma}\vec{\pi}}{F}\right]\right)^2 + \dots$$

$$= -\frac{2\ell_3 B_0^2}{F^2}(m_u + m_d)^2 \cdot \frac{\vec{\pi}^2}{2} + \frac{2\ell_7 B_0^2}{F^2}(m_u - m_d)^2 \frac{(\pi^3)^2}{2} + \dots$$

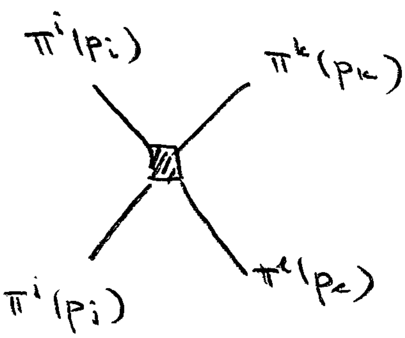
Note $m = \begin{pmatrix} m_u & \\ & m_d \end{pmatrix} = \frac{1}{2}(m_u + m_d)\,\mathbb{1} + \frac{1}{2}(m_u - m_d)\,\sigma^3$

$\pi^3 \equiv \pi^0$ is the neutral pion.

At tree level, we thus have

$$M_{\pi_i}^2 = M^2 + 2\ell_3 \frac{M^4}{F^2} - 2\ell_7 \frac{B_0(m_u - m_d)}{F^2}\,\delta_{i3}$$

At $O(M^4)$ also the one-loop diagrams from $\mathcal{L}^{(2)}$ enter. The Feynman rule for the 4-π vertex is obtained by Fourier transforming and symmetrizing



\# of contractions

$$\hat{=}\; -\frac{i}{6F^2}\cdot \underbrace{4!}_{\#\text{ of contractions}}\left[\delta^{ij}\delta^{kl}\, p_i\cdot p_k - \delta^{ij}\delta^{kl}\, p_i\cdot p_j\right] + i\frac{M^2}{F^2}\left[\delta^{ij}\delta^{kl}\right]$$

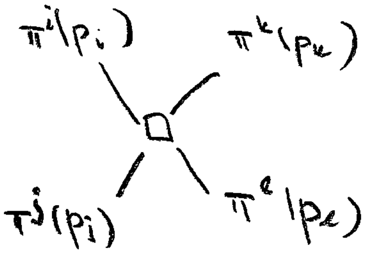
$$\hat{=}\; -\frac{i}{F^2}\Big[\delta^{ij}\delta^{kl}\left(p_i\cdot p_k + p_i\cdot p_l + p_j\cdot p_k + p_j\cdot p_l - 2p_i\cdot p_j - 2p_k\cdot p_l\right) - \delta^{ij}\delta^{kl} M^2\Big]$$

$$\hat{=}\; -\frac{i}{3F^2}\Big[\delta^{ij}\delta^{kl}\left(p_i\cdot p_k + p_i\cdot p_l + p_j\cdot p_k + p_j\cdot p_l - 2p_i\cdot p_j - 2p_k\cdot p_l - M^2\right) + \delta^{ik}\delta^{jl}\left(\text{" } j \leftrightarrow k \text{ "}\right) + \delta^{il}\delta^{jk}\left(\text{" } j \leftrightarrow l \text{ "}\right)\Big]$$

Using momentum conservation $p_i + p_j + p_k + p_\ell = 0$

and defining $s = (p_i + p_j)^2$, $t = (p_i + p_k)^2$, $u = (p_i + p_\ell)^2$

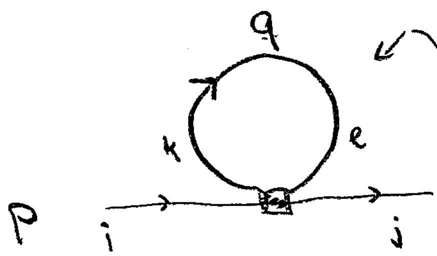
and $O_i = p_i^2 - M^2$, it can be written in the elegant form



$$= i\left[\delta^{ij}\delta^{k\ell}\frac{s-M^2}{F^2} + \delta^{ik}\delta^{j\ell}\frac{t-M^2}{F^2} + \delta^{i\ell}\delta^{jk}\frac{u-M^2}{F^2}\right]$$

$$-\frac{i}{3F^2}\left(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}\right)\left[O_i + O_j + O_k + O_\ell\right]$$

The parts involving O_i vanish on the mass shell and so do not contribute to $\pi\pi$ scattering. These parts also don't contribute to the one-loop correction to the mass, which is obtained from evaluating the self-energy



$\rightarrow \; = \dfrac{i\delta_{k\ell}}{q^2 - M^2}$

for $p^2 = M^2$*, since they either vanish or cancel the propagator denominator.

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* The physical mass M_π^2 is the value of p^2 for which $p^2 - M^2 - \underbrace{\Sigma(p^2)}_{O(M^4)} = 0$. $\rightarrow M_\pi^2 = M^2 + \Sigma(p^2 = M^2) + O(M^6)$

For the self-energy $s = 0$, $t = (p-k)^2$, $u = (p+k)^2$

$$-i\Sigma(p^2=M^2) = -\underset{\substack{\uparrow \\ \text{symmetry factor}}}{\frac{1}{2}} \frac{\delta^{ij}}{F^2} \cdot \int \frac{d^dk}{(2\pi)^d} \frac{\left(-3M^2 + \overbrace{(p-k)^2 - M^2 + (p+k)^2 - M^2}^{M^2+2pk+k^2-M^2}\right)}{k^2 - M^2}$$

$$= \delta^{ij} \frac{M^2}{2F^2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2-M^2} = -i\,\delta_{ij} \frac{M^d}{F^2} \Gamma\left(1-\frac{d}{2}\right) \cdot \frac{\pi^d}{(2\pi)^d}$$

$$= -i\,\delta_{ij} \frac{M^4}{2F^2} \left\{ 2\lambda + \frac{1}{16\pi^2} \ln\left(\frac{M^2}{\mu^2}\right) + O(d-4) \right\}$$

where $\lambda = \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2}\left(\ln(4\pi) + \underset{!}{1} - \gamma_E \right) \right\}$

The result for the pion mass to $O(M^4)$ is thus (for $m_u = m_d$)

$$M_\pi^2 = M^2 + 2 l_3 \frac{M^4}{F^2} + \frac{M^4}{F^2} \left\{ \lambda + \frac{1}{32\pi^2} \ln\left(\frac{M^2}{\mu^2}\right) \right\}$$

$$= M^2 + 2 l_3^{ren}(\mu) \frac{M^4}{F^2} + \frac{M^4}{32\pi^2} \ln\left(\frac{M^2}{\mu^2}\right)$$

where we have absorbed the divergent part λ into a renormalization of the coupling $l_3 = l_3^{ren}(\mu) - \frac{1}{2}\lambda$.

[Because of the $+1$ in the definition of λ, the scheme is not exactly $\overline{MS}$.]

there are alternative notations for ℓ_3^{ren}:

$$\ell_3^{ren} = \frac{1}{64\pi^2} \ln\left(\frac{\mu^2}{\Lambda_3^2}\right) = -\frac{1}{64\pi^2}\,\bar{\ell}_3(\mu)$$

and $\bar{\ell}_3 \equiv \bar{\ell}_3(M_\pi)$.

$$\text{so} \quad M_\pi^2 = M^2 - \frac{1}{2}\frac{M^4}{(4\pi F)^2}\underbrace{\ln\left(\frac{\Lambda_3^2}{M^2}\right)}_{\text{chiral logarithm: } \ln(m_q)} + O(M^6)$$

The expansion parameter is $\frac{M^2}{(4\pi F)^2} \approx 0{,}014$. The correction is small, of the order of a few per-cent.

A similar formula holds for F_π, the pion decay constant

$$F_\pi = F\left\{1 - \frac{M^2}{16\pi^2 F^2}\ln\frac{M^2}{\Lambda_4^2} + O(M^4)\right\}.$$

ℓ_4

Traditionally, low-energy constants such as Λ_3 and Λ_4 (or equivalently $\ell_3^{ren}, \ell_4^{ren}$) are extracted by computing a number of observables and then adjusting the low-E constants to reproduce the experimental results.

Since a few years, it becomes possible to extract such low-E constants from lattice simulations. The simulations make it possible to directly study the quark mass dependence. However, to make contact with CHPT, one needs results with relatively light dynamical fermions (see slides for numerical results).

The second application we consider is $\pi\pi$-scattering. The amplitude can be read off from the 4π Feynman rule:

$$\mathcal{A} = \delta^{ij}\delta^{kl}\,\frac{s-M^2}{F^2} + \delta^{ik}\delta^{il}\,\frac{t-M^2}{F^2} + \delta^{il}\delta^{jk}\,\frac{u-M^2}{F^2}.$$

Because of crossing symmetry, the general form is

$$A = \delta^{ij}\delta^{kl}A(s,t,u) + \delta^{ik}\delta^{il}A(t,s,u) + \delta^{ad}\delta^{bc}A(u,t,s)$$

with $A(s,t,u) = A(s,u,t)$. So our result is

$$A(s,t,u) = \frac{s-M^2}{F^2} + O(q^4) = \frac{s-M_\pi^2}{F_\pi^2} + O(q^4).$$

Instead of working with explicit isospin components i,j,k,l it makes more sense to decompose the amplitude in an isospin basis. Since the two pions are $SU_V(2)$ triplets (i.e. they have $I=1$), they can form an $I=0,1$ or 2 state, so

$$\langle I', I_3' \mid T \mid I, I_3 \rangle = T^I \delta_{II'} \delta_{I_3 I_3'}$$

The three amplitudes T^I are (Gasser & Leutwyler '84)

$$T^0 = 3A(s,t,u) + A(t,u,s) + A(u,s,t)$$
$$T^1 = A(t,u,s) - A(u,s,t)$$
$$T^2 = A(t,u,s) + A(u,s,t)$$

The scattering lengths are defined as the values of these amplitudes at threshold:

isospin

$$a_0^0 = \frac{1}{32\pi} T^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0{,}16$$

angular momentum

$$a_0^2 = \frac{1}{32\pi} T^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0{,}045$$

(Weinberg '66)

The $T^{I=1}$ amplitude vanishes at threshold because the two pions have $L=0$ and cannot form an $I=1$ state because of Bose symmetry: $\varepsilon^{ijk} |\pi^i(0)\pi^j(0)\rangle = 0$

These scattering lengths have been computed to two loops. The main theoretical uncertainty are the values of l_3 and l_4. Unfortunately $\pi\pi$ scattering is difficult to measure, in particular at low energy. Often the amplitude is extracted from the measurement of another process. Another possibility to extract the scattering lengths is to measure decays of $\pi^+\pi^-$ bound states (DIRAC exp.).
See slides for a comparison of theory and experiment.