4.2.2 Heavy flavors in QCD

The quark masses are very hierarchical and for many application one will need to integrate out the heavy flavors $m_t = 173$ GeV and $m_c = 5$ GeV. The masses of these quarks are large enough that the matching can be performed perturbatively. (For the charm $m_c = 13$ GeV, $\alpha_s(m_c) = 0.34$, this is also true, but the corrections will be significant.)

Let us first discuss left up to dimension 6 and then the matching for the $d=4$ Lagrangian.
All the operators found in QED are also allowed in QCD. The effective QCD lagrangian after integrating out the top quark has the form

\[ \mathcal{L}_{\text{d=4}} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} - \sum_{f=1}^{5} \bar{u}_f (i\gamma_5 - m_f) u_f \]

This is simply QED with 5 flavors!

For \( d = 6 \), we have operators

\[ O_{(5i)}^{5f'} = \bar{u}_f g^{[s_1} ... g^{s_{i-1}] s_1} f \bar{u}_f g^{[s_1} ... g^{s_{i-1}] s_1} f \]

\[ O_{(4i)}^{5f'} = \bar{u}_f g^{[s_1} ... g^{s_{i-1}] s_1} f \bar{u}_f g^{[s_1} ... g^{s_{i-1}] s_1} f \]

Odd numbers (even numbers suppressed)

by \( m_f \)
Since QCD does not distinguish the flavors, the Wilson coefficients of $O_{si}$ are independent of the flavor indices, i.e., we only need the two operators

$$O_{si} = \sum_{ff'} O_{si}^{ff'} ; \quad \Omega_{oi} = \sum_{ff'} \Omega_{oi}^{ff'}.$$ 

As in QED, we get

$$O_{mag} = \sum_f m_f F_{\mu\nu} G^{\mu\nu}$$

and there is one additional operator, namely

$$O_3 = \epsilon^{abc} F_{\mu\nu} F_{\rho\sigma} G^{\mu\nu} G^{\rho\sigma}.$$ 

The leading contributions to the Wilson coefficients of $O_{si}$ and $O_{mag}$ and $O_3$ arise from

* Except for the quark masses, which can be set to zero for the matching computation.
let us now discuss the matching for $L_{\text{QCD}}(4\pi)$, which contains as Wilson coefficient the coupling constant $g_s$ and $m_f$. Let us discuss the issue for the coupling constant $\alpha_s(\mu) = \frac{g_s^2(\mu)}{4\pi}$.

We denote the coupling by $\alpha_s^{\nu}(\mu)$ to distinguish $\alpha_s$ in the theory with $N_f=6$ from $\alpha_s^{\nu}(\mu)$ in the theory where the top is integrated out. Then we proceed as in our toy model. At a scale $\mu_m = m_t$ one derives a relation

$$\alpha_s^{\nu}(M_m) = \alpha_s^{6}(M_m) \frac{\beta_0^{\nu}}{\beta_0^{6}} \left[ \frac{\alpha_s^{6}(M_m)}{} \right]$$
The simplest way to obtain $\Sigma A$ is to compute the gluon propagator in both theories:

$$G_{\mu\nu} = \frac{i Z_A^0}{p^2} \left( -g_{\mu\nu} + \ldots \right)$$

Rescaling the coupling by $\frac{\alpha_s}{\bar{Z}_A}$ is the same as rescaling the gluon field. One can then show that

$$\Sigma A^{(10)} = \frac{Z_6}{Z_5}$$

where $Z_6$ is

$$Z_6 = \frac{1}{1 - \Pi^{(0)}}$$

If one chooses $\mu = m_t(\mu) m_t$ the expression for $\Sigma A$ is especially simple:

$$\Sigma A(m_t) = A + \left( \frac{13}{3} C_F - \frac{32}{9} C_A \right) T_F \left( \frac{\alpha_s(m_t)}{4\pi} \right)^2$$
Note that the QCD coupling runs differently for \( n_f = 5 \) and \( n_f = 6 \):

\[
\mu \frac{d\alpha_s(\mu)}{d \mu} = \beta(\alpha_s(\mu))
\]

\[
\beta(\alpha_s) = -2\alpha_s \left[ \beta_0 \left( \frac{\alpha_s}{4\pi} \right)^2 + \beta_1 \left( \frac{\alpha_s}{4\pi} \right)^3 + \ldots \right]
\]

with \( \beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_F \)

This running and decoupling has been implemented into a computer code RuDEC, see slides.