4.2. Decoupling of heavy flavors

4.2.1. Decoupling in QED

The quarks and leptons in the SM come in three generations. For reasons we don't understand, the masses of the fermions are quite hierarchical $m_1 \ll m_2 \ll m_3$. E.g. $m_e = 0{,}5\,\text{MeV}$, $m_\mu = 106\,\text{MeV}$, $m_\tau = 1777\,\text{MeV}$.

An important generalization of the Euler-Heisenberg EFT is the EFT obtained from integrating out heavy flavors. For QED at different energies, one uses

$$
\begin{array}{lc}
E \gtrsim m_\tau : & \mathcal{L}_{QED}[\tau, \mu, e, A] \\
 & \downarrow \\
m_\tau \gtrsim E \gtrsim m_\mu & \mathcal{L}^{\text{eff}}_{QED}[\mu, e, A] \\
 & \downarrow \\
m_\mu \gtrsim E \gtrsim m_e & \mathcal{L}^{\text{eff}}_{QED}[e, A] \\
 & \downarrow \\
m_e \gtrsim E & \mathcal{L}_{\text{Euler-Heisenberg}}[A]
\end{array}
$$

Since $m_\pi \sim m_\mu$, one needs to also consider strong interaction effects, if one includes the μ in the Lagrangian, but we'll ignore this complication for the moment and will start with $\mathcal{L}_{QED}^{eff}[\mu, e, A]$ and construct $\mathcal{L}_{QED}^{eff}[e, A]$. We'll then discuss how this Lagrangian can be used to search for new Physics and how the analogue construction works in the QCD case.

The leading power Lagrangian $\mathcal{L}_{QED}^{eff}[e, A]$ is just the QED Lagrangian $\mathcal{L} = \bar{\psi}(i\not{D} - m_e)\psi - \frac{1}{4}(F^{\mu\nu})^2$ which contains the two parameters e and m_e which are determined by matching. At higher power, we get the same photonic operators as in the Euler-Heisenberg case. In addition, there are now operators containing fermion fields. Up to operator dimension $d = 6$, we have

$$\bar{\psi}\Gamma^\mu D_\mu \psi, \quad \bar{\psi}\Gamma^{\mu\nu} D_\mu D_\nu \psi$$

$$\bar{\psi}\Gamma^{\mu\nu\rho} D_\mu D_\nu D_\rho \psi, \quad \bar{\psi}\Gamma_1 \psi \, \bar{\psi}\Gamma_2 \psi,$$

where the Γ's are arbitrary Dirac matrices.

At $d=4$, the only possibility is $\bar{\psi} i \not{D} \psi$. At $d=5$, one has

$$O_1 = \bar{\psi} \tfrac{1}{2} [\gamma^\mu, \gamma^\nu] D_\mu D_\nu \psi$$

$$= \bar{\psi} (-i) \sigma^{\mu\nu} \, i e F_{\mu\nu} \psi$$

$$= + \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi$$

$$O_2 = \bar{\psi} \overbrace{\tfrac{1}{2} \{\gamma^\mu, \gamma^\nu\}}^{g^{\mu\nu}} D_\mu D_\nu \psi$$

$O_1 + O_2 = \bar{\psi} \not{D} \not{D} \psi$ can be eliminated using the LO EOM $i\not{D}\psi = m\psi$, so we only need to consider one operator, e.g. O_1.

It turns out that the Wilson coefficient of O_1 vanishes for $m_e = 0$. The reason is that $\mathcal{L}_{QED}$ has a symmetry $\psi \to e^{i\alpha\gamma_5} \psi$, $\bar{\psi} \to \bar{\psi} e^{+i\alpha\gamma_5}$ for $m_e = 0$. and O_1 violates this symmetry. So we only need to consider the $d=6$ operator $O_{mag} = m_e \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi$.

As a side-remark, we note that the axial symmetry $\psi \to e^{i\alpha\gamma_5}\psi$ is <u>not</u> a symmetry of the theory, but only of the Lagrangian because the path-integral measure is not invariant. However, the measure for ψ the same in the full and effective theory, so that this axial anomaly does not affect our argument.

In addition, we have operators

$$O_{(n)} = \bar{\psi}\Gamma_{(n)}\psi\ \bar{\psi}\Gamma_{(n)}\psi$$

$$\Gamma_{(n)} = \gamma^{[\alpha_1}\gamma^{\alpha_2}\dots\gamma^{\alpha_n]} \qquad \text{(totally antisymmetrized)}$$

at $d=6$. Because of axial symmetry, the terms with even n will have coefficients $\propto m_e$. The operators with three covariant derivatives all reduce to $O_{(1)}$, and $O_{(3)}$. In particular

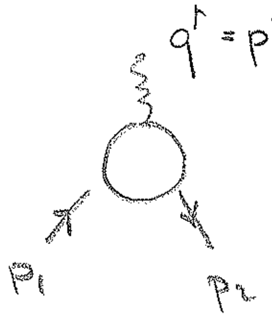
$$\bar{\psi}\,\partial_\sigma F_{\mu\nu}\,\gamma^{[\mu}\gamma^{\nu}\gamma^{\sigma]}\psi = 0$$

from the Bianchi identity.

So we conclude that, up to terms suppressed by at least $1/m_\mu^3$, all effects of physics at scales $E \gg m_e$ can be absorbed into the electron mass and the electromagnetic coupling, and the Wilson coefficients of O_{mag}, $O_{(1)}$ and $O_{(3)}$.

Let us now discuss the physics associated with O_{mag} and how the effects of physics beyond the Standard Model manifest themselves in C_{mag}.

Let's consider the interaction of an electron with a background e.m. field



$$= \bar{u}(p_2)\,\Gamma^\mu(p_1, p_2)\,u(p_1) \cdot (-ieA_\mu)$$

$$\Gamma^\mu = A\gamma^\mu + B(p_1+p_2)^\mu + C(p_1-p_2)^\mu$$
$$+ A'\gamma^\mu\gamma^5 + B'(p_1+p_2)^\mu\gamma^5 + C'\gamma^5(p_1-p_2)^\mu$$

The coefficients of A', B', C' are zero because of parity invariance of QED. Additional structures, involving $\not{p}_1$ or $\not{p}_2$ can be eliminated using the EOM $\not{p}\, u(p) = m_e u(p)$. Furthermore, current conservation implies

$$q^\mu \, \bar{u}(p_2) \Gamma_\mu u(p_1) = 0 = C \cdot q^2 \implies C = 0$$

We are thus left with two functions $A(q^2)$ and $B(q^2)$. It is customary to write Γ^μ in the form

$$\Gamma^\mu = F_1(q^2)\, \gamma^\mu - \frac{i}{2m_e} \sigma^{\mu\nu} q_\nu F_2(q^2)$$

At tree-level, we have

$$F_1(q^2) = 1$$ ← true to all orders! Definition of e.

$$F_2(q^2) = C_{mag} \cdot m_e^2$$

where we wrote

$$\mathcal{L}_{mag} = - \frac{C_{mag}}{4} O_{mag}$$

$$= - \frac{C_{mag}}{4} m_e \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi$$

Exercise: Derive the Gordon identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p+p')^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p)$$

To understand the meaning F_1 and F_2, let's consider the nonrelativistic limit $\vec{p}_1, \vec{p}_2 \to 0$. In the basis

$\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} & \sigma^i \\ \sigma^i & \end{pmatrix}$, one has

$$u(p) = \sqrt{p^0+m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\vec{p}}{p_0+m}\chi_s \end{pmatrix} \cong 2m\begin{pmatrix} \chi_s + O(p^2) \\ O(p) \end{pmatrix}$$

$$A_\mu(p_1^\mu + p_2^\mu)\frac{1}{2m}\bar{u}(p_2)u(p) = A^0\chi_s^+\chi_s + O(p^2)$$

$$A_\mu\bar{u}(p_2)\frac{i\sigma^{\mu\nu}}{2m}q_\nu u(p_1) \cong A_i\,\bar{u}(p_2)\frac{-1}{4m}[\gamma^i,\gamma^j]q_j u(p_1)$$

$$\cong 2m\cdot A_i\left(-\frac{1}{4m}\right)\chi_s^+\sigma^k\chi_s\, 2i\,\varepsilon^{ijk}q_j \qquad \left|\; \begin{matrix}[\sigma^i,\sigma^j] \\ = i\varepsilon^{ijk}\sigma^k\end{matrix}\right.$$

$$= -i\,A_i q_j\,\chi_s^+\sigma_k\chi_s\,\varepsilon^{ijk}$$

$$= \chi_s^+\vec{\sigma}\cdot\vec{B}(q)\chi_s$$

where $B_k(q) = -i\varepsilon^{ijk}q_iA_j(q) \hat{=} \vec{\nabla}\times\vec{A}(q)$

The QM Hamiltonian describing the interaction of an electron with an e.m. field contains a term

$$\mathcal{H} = -\underbrace{g_e \frac{e}{2m} \vec{S}}\cdot\vec{B}$$

$$= -\vec{\mu}\cdot\vec{B}$$

For an electron $\vec{S} = \frac{\vec{\sigma}}{2}$, and comparing with our expression for $\bar{u}\Gamma^\mu u\,(-ieA_\mu)$, we find

$$g_e = 2\left[F_1(0) + F_2(0)\right]$$

$$= 2 + 2F_2(0)$$

The deviation of the gyromagnetic ratio g_e from two is called the anomalous magnetic moment

$$a_e = \frac{g_e - 2}{2} = F_2(0)$$

It receives contributions from quantum corrections and from the operator O_{mag}, whose Wilson coefficient encapsulates the contribution from heavier states.

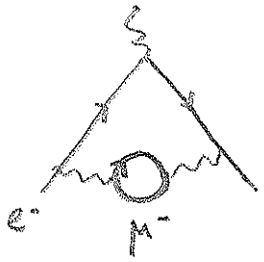
Because it is sensitive to corrections from heavier states, precision measurements of anomalous magnetic moments are used to search for New Physics.

Let's look at contributions to a_e:

$\longrightarrow \quad \frac{\alpha}{2\pi}$ ($\alpha = \frac{e^2}{4\pi}$) (Schwinger '48)

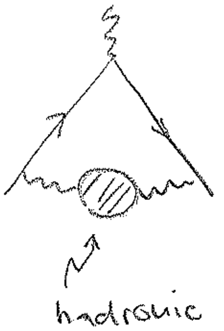
$= 10^{-3}$

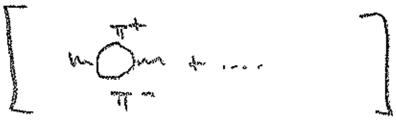
(α^4 correction is known!)



$$\longrightarrow \quad -\frac{32}{45}\frac{m_e^2}{m_\mu^2}\left(\frac{\alpha}{2\pi}\right) = m_e^2\, \Delta C_{mag}$$

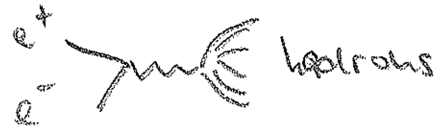
(see e.g. 0908.4392)



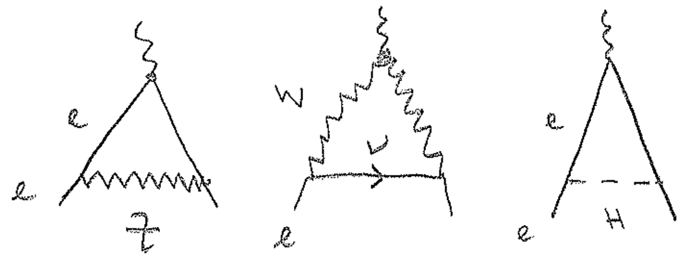


$\longrightarrow \quad \Delta C_{mag}^{\text{Hadronic}}\, m_e^2$

obtained experimentally from measuring

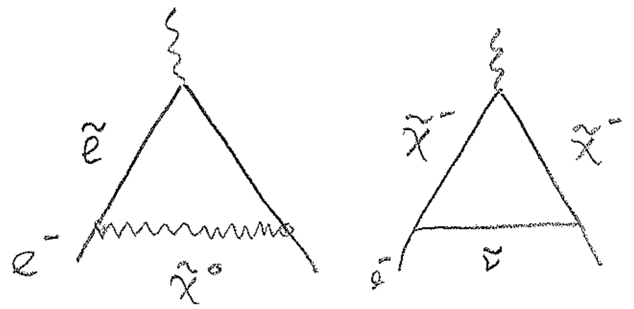


Electroweak corrections



$\propto \frac{\alpha}{\pi} \frac{m_e^2}{m_W^2}$

SUSY



$\propto \frac{\alpha}{\pi} \frac{m_e^2}{\tilde{m}^2}$

→ If the SUSY scale $\tilde{m}$ is not much larger than m_W, the SUSY effects can compete with EW effects.

Since $(m_e/m_W)^2 \sim 4 \cdot 10^{-11}$, the weak effects are very small for the electron $g-2$. Also, the $g-2$ measurement is used to determine α, so it is not possible to search for new physics (unless α is extracted differently). The effect of higher-mass particles are enhanced by $\left(\frac{m_\mu}{m_e}\right)^2 \approx 40$ in the muon $g-2$. This quantity was measured very precisely at

Brookhaven (see slides for an explanation of the expeiment and results). Interestingly, the measurement deviates at the 3σ level from the theoretical prediction. This could indicate the presence of new particles (Marciano: "fits susy like a glove") or could be due to a problem in the determination of the hadronic corrections.