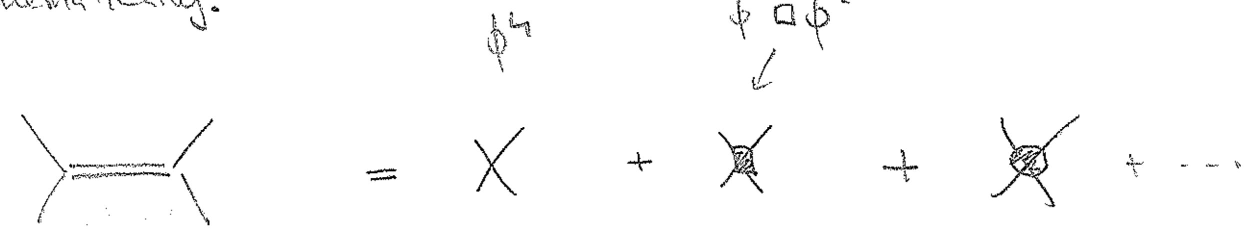
3.3. Matching at higher orders

The general method to perform the matching is always the same: compute the same quantity both in the EFT and in the full theory and adjust the Wilson coefficients ("coupling constants") in the EFT such that the results agree to the given order of the low energy expansion.

At tree level, it is simple to understand that the procedure works: the heavy particles are always far off-shell and their propagators get expanded into a polynomial. Since $\mathcal{L}_{eff}$

contains all higher-dimensional operators, it generates the most general polynomial contribution to the amplitude and adjusting the couplings, we reproduce the full theory result.

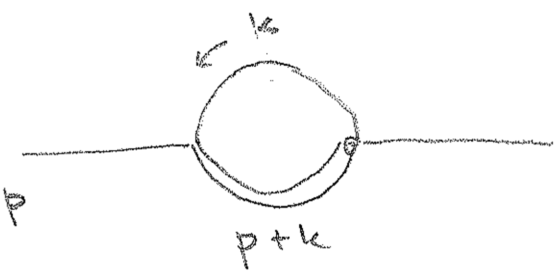
Schematically:



$$\frac{1}{p^2 - M^2} = -\frac{1}{M^2} - \frac{1}{M^2}\frac{p^2}{M^2} - \cdots$$

At higher orders, there are several interesting new features.

Consider the full theory diagram



* The loop momentum k can be small or large: heavy propagator is <u>not</u> always off shell.

* Renormalization: there are UV divergences both in the full and effective theory.
  -> renormalize all couplings / Wilson coeffs.

* Loop diagrams are nontrivial (i.e. non-analytic) function of external momenta and masses, but Wilson coefficients C_i can only depend on high-energy scale M.
  ($\mathcal{L}_{eff}$ must be local.)

  -> The nontrivial part of the amplitudes must be present in EFT computation.
  (cannot be obtained from C_i)

This works out ~~because~~ the low-E dynamics of $\mathcal{L}_{full}$ and $\mathcal{L}_{eff}$ is the same, but it will be nice to see this in an example.

* Due to renormalization in $\mathcal{L}_{eff}$, the Wilson coefficients depend on the renormalization scale μ:

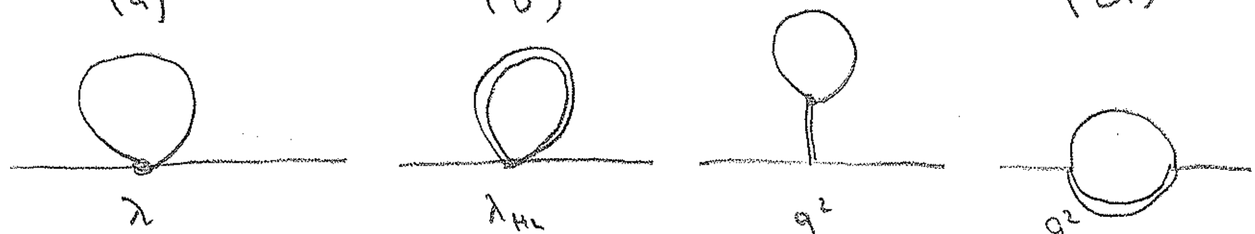
$$C_i = C_i(\mu).$$

The coefficients involve terms $g^2 \ln\left(\frac{\mu}{M}\right)$, $\lambda_{HL} \ln\left(\frac{\mu}{M}\right)$ etc.

To see how all this works in practice, we will now perform a one-loop matching computation in the simplest possible setting: We will compute the two-point function and will compute the one-loop corrections to $\tilde{m}$, the mass of the light particle in $\mathcal{L}_{eft}$. The diagrams are:

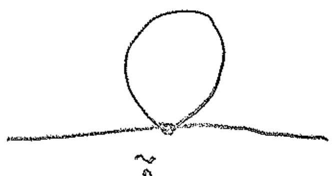
Full theory:

(a) (b) (c) (d)



λ λ_{HL} g^2 g^2

EFT:



$\tilde{\lambda}$

Diagram (a):  $= -i\Sigma$

$$\Sigma^{(a)} = i(-i\lambda_L)\,\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2}\times\mu^{2\varepsilon}$$

symmetry factor

renormalization scale, introduced to make λ dimensionless in $d = 4-2\varepsilon$.

$$= \frac{\lambda_L}{2}\,(4\pi)^{-\frac{d}{2}}\,\Gamma\left(1-\frac{d}{2}\right)(m^2)^{d/2-1}\,\mu^{2\varepsilon}$$

see Appendix A

$$= \frac{m^2\lambda_L}{32\pi^2}\cdot\left[-\frac{1}{\varepsilon}+\gamma_E-\ln(4\pi)-1+\ln\left(\frac{m^2}{\mu^2}\right) + O(\varepsilon)\right]$$

Since $-\gamma_E+\ln(4\pi)$ always appear together with the $1/\varepsilon$ term, and since they are numerically not small, it is customary to remove not only the $\frac{1}{\varepsilon}$-terms ($\equiv$ MS scheme) but the entire combination ($\equiv \overline{\text{MS}}$ scheme) by renormalization. One way to achieve this, is to set

$\bar{\mu}^2 = \mu^2 e^{\gamma_E}\,4\pi$ so that

$$\Sigma^{(a)} = \frac{\lambda_L}{32\pi^2}\,m^2\left[-\frac{1}{\varepsilon}-1+\ln\left(\frac{m^2}{\bar{\mu}^2}\right)+O(\varepsilon)\right]$$

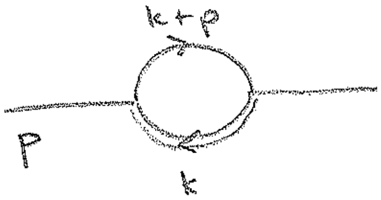
Diagram (b)  : $\Sigma^{(b)} = \frac{\lambda_{HL}}{32\pi^2} M^2 \left[-\frac{1}{\varepsilon} - 1 + \ln\left(\frac{M^2}{\mu^2}\right) \right]$

Diagram (c) 

$$\Sigma^{(c)} = i \, \frac{1}{2} (-ig)^2 \, \frac{i}{-M^2} \int \frac{d^dk}{(2\pi)^d} \, \frac{i}{k^2 - m^2} \, \mu^{2\varepsilon}$$

$$= \frac{g^2}{32\pi^2} \, \frac{m^2}{M^2} \left[+\frac{1}{\varepsilon} + 1 - \ln\left(\frac{m^2}{\mu^2}\right) \right]$$

The only non-trivial contribution arises from (d)

$\Sigma^{(d)} =$ 

$= i \, (ig^2) \int \frac{d^dk}{(2\pi)^d} \, \frac{i^2 \, \mu^{2\varepsilon}}{[(k+p)^2 - m^2](k^2 - M^2)}$

The simplest way to evaluate $i\Sigma^{(d)}(p^2, m^2, M^2)$ is to take $i\Sigma^{(d)}(0, 0, M^2)$ from the master formula for d-dim loop integrals in Appendix A and then to evaluate the finite difference

$$\Delta\Sigma = \Sigma^{(d)}(p^2, m^2, M^2) - \Sigma^{(d)}(0, 0, M^2)$$

in $d = 4$.

This yields

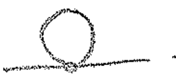
$$\Sigma^{(a)} = \Sigma^{(M)}(0,0,M^2) + \Delta\Sigma(p^2, m^2, M^2)$$

$$= \frac{g^2}{16\pi^2}\left[-\frac{1}{\varepsilon} - 1 + \ln\left(\frac{M^2}{\mu^2}\right)\right]$$

$$+ \frac{g^2}{16\pi^2}\left[-\frac{p^2}{2M^2} - \frac{m^2}{M^2}\ln\left(\frac{m^2}{M^2}\right) + O\left(\frac{1}{M^4}\right)\right]$$

were I have expanded $\Delta\Sigma$ in $\frac{p^2}{M^2}$ and $\frac{m^2}{M^2}$.

Finally, the effective theory diagram is

$$\Sigma = $$  $$= \frac{\tilde{\lambda}}{8\pi^2} m^2 \left(-\frac{1}{\varepsilon} - 1 + \ln\left(\frac{m^2}{\mu^2}\right)\right)$$

We now renormalize, i.e. absorb the $\frac{1}{\varepsilon}$-pieces of Σ into the couplings of the full and effective theory and then calculate the difference

$$\Delta = \Sigma_{full} - \Sigma_{eff}$$

and absorb it into the couplings of $\mathcal{L}_{eff}$.

Since we will end up also with a wave-function renormalization, we write the effective theory Lagrangian as

$$\mathcal{L}_{eff} = \frac{1}{2} Z (\partial_\mu \phi)^2 - \frac{\tilde{m}^2}{2} Z \phi^2 - \frac{\tilde{\lambda}}{4!} Z^2 \phi^4$$

with $Z = 1 + O(\lambda)$. (The one-loop Z-factor in the full theory happens to be trivial.)

Then

$$0 \stackrel{!}{=} \Delta = m^2 - p^2 + Z\tilde{m}^2 - Zp^2 + \Sigma_{full}^{(loop)} - \Sigma_{eff}^{(Loop)}.$$

In order for this to work, all low-energy physics has to drop out of Δ. In particular, the $\log(m)$ pieces in Σ_{full} and Σ_{eff} have to cancel!

Let's look at these terms

$$\frac{16\pi^2}{m^2}\Delta = \ln(m)\left[\lambda_L - \frac{3g^2}{M^2} - \tilde{\lambda} \right] + \dots$$

In our tree-level calculation we found $\tilde{\lambda} = \lambda_L - \frac{3g^2}{M^2}$, so our $\mathcal{L}_{eff}$ indeed reproduces the low-E part of the full theory.

The remaining terms are

$$\Delta = m^2 - Z\tilde{m}^2 + \frac{1}{16\pi^2}\left(g^2\left(1+\frac{m^2}{M^2}\right) + \frac{M^2}{2}\lambda_{HL} \right)\left(\ln\left(\frac{M^2}{\mu^2}\right)-1\right)$$

$$- p^2 + Z p^2 - \frac{g^2}{32\pi^2} p^2$$

$$\Longrightarrow \quad Z = 1 + \frac{g^2}{32\pi^2}$$

$$\tilde{m}^2 = m^2\left(1 - \frac{g^2}{32\pi^2}\frac{1}{M^2}\right) + \left(g^2\left(1+\frac{m^2}{M^2}\right)+\frac{M^2}{2}\lambda_{HL}\right)\frac{\ln\left(\frac{M^2}{\mu^2}\right)-1}{16\pi^2}$$

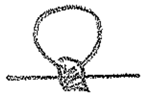
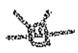
$$+ O(\lambda, g)$$

Note the presence of the $M^2\lambda_{HL}$ contribution.

The same contributions will also arise in the physical mass m_{ph} determined by $\Sigma(p^2 = m_{ph}^2) = 0$.

If m_{ph} is small, this implies a large cancellation.

This is again the statement that small scalar masses are unnatural.

Expanding the full theory result to higher power in p^2, one can determine the Wilson coefficients of operators $\frac{1}{2} C_{(2,2n)} \phi_L \Box^n \phi_L$. Note that in the matching for $\phi_L \Box^2 \phi_L$, a new effective theory diagram , where  $\propto \phi_L^2 \Box \phi_L^2$, contributes.

By computing the four and six-point functions one can then determine also $\tilde{\lambda}$, $C_{(4,2)}$ and $C_{(6)}$ to one-loop accuracy. In this case, the number of diagrams becomes quite large. On the other hand, for $\tilde{\lambda}$, the four-point function at vanishing momentum is sufficient.

3.4 Power counting

In our matching calculation we have assumed that the higher order Lagrangians do not contribute at leading power. At the tree level, this is obvious, since the operators in the power suppressed Lagrangians have additional derivatives and or fields. But in loop integrals the momenta are large, so it's not trivial that higher derivative terms are suppressed.

E.g. the contribution of $\frac{1}{(M^2)^n} \phi_L^2 \, \Box^n \, \phi_L^2$ to the two point function at zero external momentum is

$$\delta \Sigma \propto \frac{1}{(M^2)^n} \int d^dk \, \frac{(k^2)^n}{(k^2 - m^2)} = \, ?$$

The beauty of dim. reg. is that the loop integrals in the EFT only depend on low-energy scales, therefore by dimensional analysis $\delta \Sigma \propto (m^2)^{d/2-1} \times (m^2)^n \cdot \frac{1}{(M^2)^n}$

So the loop graph is indeed suppressed by $\left(\frac{m^2}{M^2}\right)^n$.

Note that in cut-off regularization

$$\frac{1}{(M^2)^n} \int^{\Lambda} d^4k \; \frac{(k^2)^n}{k^2 - m^2} \;\; \propto \;\; \Lambda^{d-2} \times \Lambda^{2n} / M^{2n} + \ldots$$

the loop contributions of higher-dim operators are unsuppressed. The terms which violate the tree level power counting are trivial cut-off terms but they make computations cumbersome.

In contrast, the power counting in dim. reg is very simple. To get a quantity up to $\left(\frac{1}{M^2}\right)^n$ accuracy, we need the Lagrangian up to $\left(\frac{1}{M^2}\right)^n$ and the $\left(\frac{1}{M^2}\right)^n$ corrections arise from diagrams with:

a single vertex from $\mathcal{L}_{eff}^{(n)}$,

or one from $\mathcal{L}_{eff}^{(n-m)}$ and one from $\mathcal{L}^{(m)}$,

ore one from $\mathcal{L}_{eff}^{(n-m_1-m_2)}$ and one from $\mathcal{L}^{(m_1)}$ and $\mathcal{L}^{(m_2)}$,

etc.