May 28, 2021

Effective Field Theory

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Abstract

This lecture provides an introduction to the framework of low-energy effective field theories. After developing the basic concepts, the method is used to analyze electromagnetic, weak, and strong interactions at low energies. The course is intended for master or graduate students who have taken a first course in quantum field theory.

Contents

1. Introduction

Effective field theory was first developed in the context of the strong interactions [1–3], but has since become an important tool in all of particle and nuclear physics (and beyond). It is based on ideas related to the Wilsonian renormalization group [4], which describes the evolution of operators as a function of the renormalization scale. In the first two chapters we will review its basics and introduce the concepts and techniques of effective field theory using the example of scalar fields. The later chapters then address applications, mainly to the low-energy properties of the Standard Model.

These notes were compiled for the course "Effective Field Theory" by Thomas Becher delivered at the University of Bern in 2010 and 2015, and set in LaTeXby Jonas Haldemann. The present version has been revised and corrected by Martin Hoferichter for a course in 2021. The material covered is largely based on Refs. [5–8], to which we also refer for further reading. Of course, we take full responsibility for all typos and mistakes introduced or overlooked by us.

2. The Wilsonian effective action

Consider a field theory with characteristic large energy scale M, and suppose we are only interested in physics at low energies $E \ll M$. This is the physical situation effective field theories are designed to analyze. The full theory is defined in terms of a path integral. Everything we wish to know can be obtained from calculating the expectation values

$$\langle 0| T\{\phi(x_1)\dots\phi(x_n)\}|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, e^{iS(\phi)}\phi(x_1)\dots\phi(x_n), \tag{2.1}$$

where the integration measure is

$$\int \mathcal{D}\phi = \prod_{x^\mu}\int d\phi(x) \quad \text{or} \quad \int \mathcal{D}\phi = \prod_{p^\mu}\int d\tilde{\phi}(p), \tag{2.2}$$

and

$$Z = \int \mathcal{D}\phi e^{iS(\phi)}. \tag{2.3}$$

To obtain the low-energy effective action, we split the field

$$\phi = \phi_L + \phi_H, \qquad \int \mathcal{D}\phi = \int \mathcal{D}\phi_L \int \mathcal{D}\phi_H, \tag{2.4}$$

where ϕ_H contains all Fourier modes with $\omega \geq \Lambda$ and ϕ_L the low-energy modes $\omega < \Lambda$. Since we are only interested in low-energy physics, we only need to consider correlation functions

$$\begin{aligned}\langle 0| T\{\phi_L(x_1)\dots\phi_L(x_n)\}|0\rangle &= \frac{1}{Z}\int \mathcal{D}\phi_L \underbrace{\int \mathcal{D}\phi_H\, e^{iS(\phi_L+\phi_H)}}_{e^{iS_\Lambda(\phi_L)}} \phi_L(x_1)\dots\phi_L(x_n) &(2.5)\\ &= \frac{1}{Z}\int d\phi_L\, e^{iS_\Lambda(\phi_L)}\phi_L(x_1)\dots\phi_L(x_n). &(2.6)\end{aligned}$$

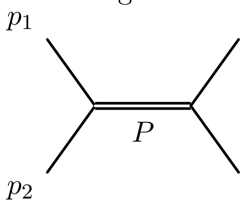
$S_\Lambda(\phi_L)$ is called the "Wilsonian effective action" and we have chosen $\Lambda \leq M$ to integrate out the physics associated with M. $S_\Lambda(\phi_L)$ is non-local on scales $\Delta x^\mu \gtrsim \frac{1}{\Lambda}$ (i.e., the Lagrangian is not just a polynomial of the fields or their derivatives evaluated at a single point in space-time), because high-energy fluctuations have been integrated out. As a final step one expands the non-local action as a series of local operators. This expansion is possible because $E \ll \Lambda$. The result has the form

$$S_\Lambda(\phi_L) = \int d^dx\, \mathcal{L}_\Lambda^{\text{eff}}(x), \tag{2.7}$$

$$\mathcal{L}_\Lambda^{\text{eff}}(x) = \sum_i g_i O_i(x), \tag{2.8}$$

where the object $\mathcal{L}_\Lambda^{\text{eff}}$ is called the effective Lagrangian. It is an infinite sum over local operators O_i allowed by symmetries. The coefficients g_i are referred to as Wilson coefficients.

To make this a little more concrete, assume that we integrated out a heavy particle with mass M. The full theory might contain diagrams such as



$$\sim \frac{1}{P^2 - M^2}, \tag{2.9}$$

where the two incoming lines represent the light field ϕ_L and the double line the heavy field ϕ_H. Since $p_1, p_2 \ll M$, we can expand

$$\frac{1}{p^2 - M^2} = -\frac{1}{M^2} - \frac{p^2}{M^4} + \cdots \to -\frac{1}{M^2}\delta^{(4)}(x) + \frac{\Box}{M^4}\delta^{(4)}(x), \tag{2.10}$$

so $\mathcal{L}_\Lambda^{\text{eff}}$ will contain terms such as $\phi_L^4(x)$ and $\partial_\mu \phi_L(x) \partial^\mu \phi_L(x) \phi_L^2(x)$ etc. In general it will be very hard to calculate the coefficients g_i and since we ended up with infinitely many terms in $\mathcal{L}_\Lambda^{\text{eff}}$ it is, a priori, unclear how the construction is useful.

The required ordering principle is provided by dimensional analysis. With $\hbar = c = 1$ $[m] = [E] = [x^{-1}] = [t^{-1}]$ all quantities measured in the same units. Assuming that $[g_i] = -\gamma_i$ is the mass dimension of g_i, it follows that

$$g_i = C_i M^{-\gamma_i}, \tag{2.11}$$

with a dimensionless coefficient C_i. Since the coefficients arose when integrating out the physics associated with M, it is natural to assume that $C_i = \mathcal{O}(1)$. Very large, e.g., $C_i \sim 10^6$, or very small coefficients, e.g., $C_i \sim 10^{-6}$, would call for some explanation in terms of degrees of freedom not considered. At low energy, the contribution of O_i to a dimensionless observable scales as

$$\left(\frac{E}{M}\right)^{\gamma_i} = \begin{cases} \mathcal{O}(1) & \gamma_i = 0 \\ \gg 1 & \gamma_i < 0 \\ \ll 1 & \gamma_i > 0 \end{cases}, \tag{2.12}$$

and therefore only operators with $\gamma_i \le 0$ are important at low energy.

To derive the mass dimension δ_i of a given operator, we need to know the mass dimension of the fields. Assuming that the theory is weakly coupled, the scaling dimension is determined by the free action

$$S_0 = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2\right) \tag{2.13}$$

with $[x] = -1$, $[\partial_\mu] = +1 \sim E$, and using that the action is dimensionless, we find

$$[\phi] = \frac{d}{2} - 1, \qquad \text{i.e.} \quad \phi \sim E^{\frac{d}{2}-1}. \tag{2.14}$$

For an operator with mass dimension δ_i, we have $\gamma_i = \delta_i - d$, e.g.

	δ_i	γ_i	scaling of g_i
$\partial_\mu \phi \partial^\mu \phi$	d	0	1
ϕ^2	$d-2$	-2	M^2
ϕ^4	$2d-4$	$d-4$	M^{4-d}
$(\partial_\mu \phi)^2 \phi^2$	$2d-2$	$d-2$	M^{2-d}
ϕ^6	$3d-6$	$2d-6$	M^{6-2d}

For an operator with n scalar fields and m derivatives we have

$$\delta_i = n\left(\frac{d}{2} - 1\right) + m, \qquad \gamma_i = (n-2)\left(\frac{d}{2} - 1\right) + m - 2, \tag{2.15}$$

so only very few operators have $\gamma_i \leq 0$ (unless $d \leq 2$). The following terminology is commonly used:

Dimension	Importance for $E \to 0$	Terminology for operator
$\delta_i < d \quad \gamma_i < 0$	grows	relevant (super-renormalizable)
$\delta_i = d \quad \gamma_i = 0$	constant	marginal (renormalizable)
$\delta_i > d \quad \gamma_i > 0$	declines	irrelevant (non-renormalizable)

This terminology is not optimal. For example, it is interesting to search for the effects mediated by irrelevant operators, since they provide information on the physics at very high energies.

Moreover, our discussion makes it clear that renormalizability is sometimes overrated: usually one avoids irrelevant operators because they render a theory nonrenormalizable. However, once we admit that a theory is not valid up to infinitely large energies, then it is clear that it will contain also irrelevant operators. This is not a problem, because their contributions are suppressed by some large scale M, at which new physics enters. Renormalizable Lagrangians are so successful in describing our low-energy measurements because the relevant and marginal operators are the most important ones at low energies.

2.1. Examples of irrelevant operators

Before we proceed further with the renormalization group, we give some examples of irrelevant operators

- The gauge symmetries of the Standard Model allow us to write down a dimension-5 term $g\,O = g\,\nu^T H H \nu$ with $g \sim \frac{1}{\Lambda}$ (ν and H are neutrino and Higgs fields, respectively). After electroweak symmetry breaking this yields a Majorana mass term for the neutrino, with $m_\nu \sim \frac{\langle H \rangle^2}{\Lambda}$ (where $\langle H \rangle \sim 174\,\mathrm{GeV}$ is the vacuum expectation value of the Higgs field). The fact that $\Lambda \sim 10^{14}$ GeV can be interpreted as evidence for physics beyond the Standard Model at these scales.

- The weak interaction is so weak at low energies because it is mediated by irrelevant operators such as $O = \bar{u}\gamma_\mu(1-\gamma_5)d\,\bar{l}\gamma^\mu(1-\gamma_5)\nu$. The fermion field scales as $\psi \sim E^{\frac{3}{2}}$, so $\delta = 6$, $\gamma = 2$. The coefficient of the operator must be proportional to $\frac{1}{M^2}$. Here, the mass $M = M_W$, is the mass of the W-boson. From the form of the interaction, Oskar Klein predicted the existence of massive particles with $M_W \geq 60$ GeV already in 1938.

While irrelevant operators are perfectly natural, super-renormalizable/relevant operators are problematic. Consider for example the ϕ^2 operator in ϕ^4 theory. In $d = 4$, we have $\delta_i = 2$, $\gamma_i = -2$, and so we expect that $m^2 \sim \Lambda^2$. Integrating out the quantum fluctuations at large scales generates a large mass for scalar particles. But this is a contradiction: if $m^2 \sim \Lambda^2$ we should have integrated out the corresponding field ϕ. Note that also $\bar{\psi}\psi \sim E^3$ is relevant. This reasoning leads one to conclude that only theories whose mass terms are forbidden by symmetries are natural. Looking at the Standard Model as an effective theory, this condition is almost fulfilled:

- gauge bosons do not have mass terms because they are forbidden by gauge symmetry, i.e.
$$m^2(A_\mu)^2 \to m^2 e^{2i\alpha}(A_\mu)^2 \tag{2.16}$$
is not invariant,
- fermion fields do not have mass terms because left- and right-handed fields
$$\psi_L = \frac{1}{2}(1-\gamma_5)\psi, \tag{2.17}$$
$$\psi_R = \frac{1}{2}(1+\gamma_5)\psi, \tag{2.18}$$
have different gauge charges: ψ_R is neutral under $SU(2)_L$, ψ_L is not, so that a mass term $m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$ would violate gauge invariance.

Note that the absence of a mass term in $\mathcal{L}_{\mathrm{SM}}$ does not imply that the fermions and gauge bosons are massless. They receive their mass by interacting with the Higgs condensate $\bar{\psi}_L H \psi_R \to \bar{\psi}_L \langle H\rangle \psi_R$, where $\langle H\rangle$ is the vacuum expectation value. The only mass term in the Standard Model is the mass term of the Higgs field $\mu^2 H^\dagger H$. There are several ways out of this dilemma, but all of them involve physics beyond the Standard Model around the scale of the Higgs mass:

- Supersymmetry relates fermions and bosons. It can be used to protect scalar masses. Constructing the theory in such a way that fermion masses are forbidden implies that scalar masses are forbidden as well.
- In technicolor models, the Higgs boson is a bound state of a fermion–antifermion pair, similar to mesons in QCD.
- In little Higgs models, the Higgs boson is a pseudo Goldstone boson of a spontaneously broken global symmetry.

Alternatively, the smallness of M_H could just be due to some accidental cancellation. To make this more plausible, people often invoke the anthropic principle: "if the Universe (in our example M_H) were much different, we would not be here." There is no concrete evidence for any of these explanations, and we will not revisit this more philosophical discussion in the following.

2.2. Renormalization Group

So far, we have considered a situation where we integrated out physics above some characteristic scale M. It is also interesting to look at what happens if we only integrate out a small slice $\Lambda > \omega > \Lambda - \delta\Lambda$ in which the particle content remains unchanged. In this case the form of the action is unchanged, only the coefficients g_i change. Repeating the procedure, one obtains the couplings as a function of the cutoff
$$\{g_i(\Lambda)\} \to \{g_i(\Lambda - \delta\Lambda)\} \to \{g_i(\Lambda - 2\delta\Lambda)\} \to \dots \tag{2.19}$$

The evolution equations

$$\Lambda \frac{dg_i}{d\Lambda} = f(\{g_i\}) \tag{2.20}$$

are called renormalization group (RG) equations (the transformation of the theory from one scale to another can be formally interpreted as a group transformation in the mathematical sense). Let us derive this RG evolution for the trivial but instructive case of a quadratic action. The general form is

$$\mathcal{L} = \frac{1}{2}\phi(x)[-m^2 - 1 \times \Box + c\Box^2 + \ldots]\phi(x). \tag{2.21}$$

After Fourier transform

$$\phi(x) = \int_k e^{-ikx}\tilde{\phi}(k) = \int \frac{d^d k}{(2\pi)^d} e^{-ikx}\tilde{\phi}(k), \tag{2.22}$$

the action becomes

$$\begin{aligned} S &= \frac{1}{2}\int d^d x \int_p \int_k \tilde{\phi}(p)[-m^2 + k^2 + ck^4 + \ldots]\tilde{\phi}(k)e^{-i(p+k)x} && (2.23) \\ &= \frac{1}{2}\int_k \tilde{\phi}(-k)[-m^2 + k^2 + ck^4 + \ldots]\tilde{\phi}(k). && (2.24) \end{aligned}$$

We further assume that our theory is defined with a UV cutoff Λ:

$$\int_k \to \int_k^\Lambda = \int_{-\Lambda}^{\Lambda} \frac{dk^0}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk^1}{2\pi} \cdots \int_{-\Lambda}^{\Lambda} \frac{dk^{d-1}}{2\pi}. \tag{2.25}$$

Let us now split $\phi = \phi_L + \phi_H$ according to

$$\begin{aligned} \tilde{\phi}(k) &= \tilde{\phi}_L(k) + \tilde{\phi}_H(k) && (2.26) \\ &= \begin{cases} \tilde{\phi}_L(k) & |k_\mu| < b\Lambda, \quad \forall \mu \\ \tilde{\phi}_H(k) & |k_\mu| > b\Lambda, \quad \text{for some } \mu \end{cases} && (2.27) \end{aligned}$$

The field ϕ_L describes functions below $\Lambda' = b\Lambda$, with a free parameter $b \in [0, 1]$. Our action splits accordingly into

$$S = S_L + S_H = \frac{1}{2}\int_k^{b\Lambda} \tilde{\phi}_L(-k)[\ldots]\tilde{\phi}_L(k) + \frac{1}{2}\int_k^{\Lambda} \tilde{\phi}_H(-k)[\ldots]\tilde{\phi}_H(k). \tag{2.28}$$

If we are only interested in low energy Green's functions

$$\langle 0| T\{\phi_L(x_1)\ldots\phi_L(x_n)\} |0\rangle = \frac{1}{Z}\int \mathcal{D}\phi_L \int \mathcal{D}\phi_H \, e^{iS_H} e^{iS_L} \phi_L(x_1)\ldots\phi_L(x_n), \tag{2.29}$$

then the effect of ϕ_H is absorbed by the normalization, so that

$$\langle 0| T\{\phi_L(x_1)\ldots\phi_L(x_n)\} |0\rangle = \frac{1}{Z_L}\int \mathcal{D}\phi_L \, e^{iS_L} \phi_L(x_1)\ldots\phi_L(x_n). \tag{2.30}$$

To compare S_L with S, let us now rescale

$$k' = \frac{k}{b}, \qquad x' = xb. \tag{2.31}$$

In terms of the variable k', the cutoff moves back to Λ. The action becomes

$$S_L = \frac{1}{2}\int_{k'}^{\Lambda} b^d \tilde{\phi}(-bk')[-m^2 + b^2 k'^2 + b^4 c k'^4 + \ldots]\tilde{\phi}(bk'). \tag{2.32}$$

Let us further rescale $\tilde{\phi}(bk') \to \tilde{\phi}'(k') \times b^{-\frac{d+2}{2}}$ to have a canonically normalized kinetic term. The resulting action

$$S_L = \frac{1}{2}\int_{k'}^{\Lambda} \tilde{\phi}'(-k')[-\frac{m^2}{b^2} + k'^2 + b^2 c k'^4 + \ldots]\tilde{\phi}'(k') \tag{2.33}$$

shows that we get the same theory, but with

$$m^2 \to \frac{m^2}{b^2} \quad \text{(relevant)}, \qquad c \to b^2 c \quad \text{(irrelevant)}, \tag{2.34}$$

so for $b = \frac{1}{2}$, for example, the mass becomes four times as large, while the coefficient of the four-derivative term is four times smaller. If we iterate the transformation (making b smaller and smaller), we get the *renormalization group flow* in the space of coupling constants:

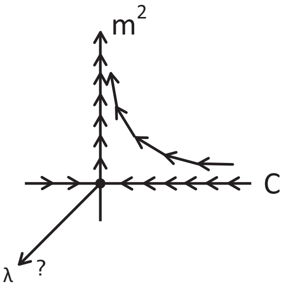


Figure 2.1.: The point $m = c = \cdots = 0$, the massless scalar field action, is a *fixed point*. This is called the Gaussian fixed point.

When this analysis is extended to theories that include small couplings, the result is basically unchanged. The irrelevant operators remain irrelevant, and the relevant ones stay relevant. However, it becomes very interesting to check what happens with *marginal* operators. The small perturbation induced by the coupling will make them marginally relevant, or marginally irrelevant. For QCD, it turns out that the coupling slowly gets stronger as the high-energy modes are integrated out. Starting with an essentially free theory defined with a very high cutoff Λ, one ends up with a strongly coupled theory at low energy. This property is called asymptotic freedom. As we will show now, the situation is opposite for ϕ^4 theory. Even if the theory has a large coupling in its Lagrangian, it looks more and more like a free theory when the high-energy modes are integrated out.

We now turn to ϕ^4 theory with a cutoff Λ

$$Z = \int \mathcal{D}\phi \exp\left[- \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right)\right], \tag{2.35}$$

where $\mathcal{D}\phi = \prod_{|k|<\Lambda} d\tilde{\phi}(k)$. Moreover, we switched to Euclidean space[1] to facilitate isolating a given momentum region in the integral, i.e., $t_M = -it_E$, $x_M^2 = -x_E^2$, in the above expression $x^\mu \equiv x_E^\mu$, $k \equiv k_E^\mu$. Let us now again split

$$\tilde{\phi}(k) = \tilde{\phi}_L(k) + \tilde{\phi}_H(k), \tag{2.36}$$

$$\tilde{\phi}_H(k) = \tilde{\phi}(k)\Xi(k), \tag{2.37}$$

$$\Xi(k) = \Theta(|k| < \Lambda)\Theta(|k| > b\Lambda). \tag{2.38}$$

The quadratic part of the action will again just turn into a sum of quadratic actions, but the interaction now also includes crossed terms:

$$S(\phi_L + \phi_H) = S(\phi_L) + S(\phi_H) + \int d^4x\, \lambda \left[\frac{\phi_L\phi_H^3}{3!} + \frac{\phi_L^2\phi_H^2}{2!2!} + \frac{\phi_L^3\phi_H}{3!}\right]. \tag{2.39}$$

Now we will derive the Feynman rules and then integrate over ϕ_H to lowest order in perturbation theory. The propagators in the theory are

$$\Delta_L = \langle 0| T\{\phi_L(x)\phi_L(0)\} |0\rangle = \int^\Lambda \frac{d^4k}{(2\pi)^4} e^{ikx} \frac{1}{k^2+m^2}\Theta(|k| < b\Lambda), \tag{2.40}$$

$$\Delta_H = \langle 0| T\{\phi_H(x)\phi_H(x)\} |0\rangle = \int^\Lambda \frac{d^4k}{(2\pi)^4} e^{ikx} \frac{1}{k^2+m^2}\Theta(|k| > b\Lambda), \tag{2.41}$$

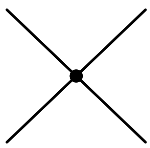
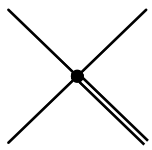
$$\langle 0| T\{\phi_H(x)\phi_L(x)\} |0\rangle = 0, \tag{2.42}$$

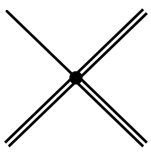
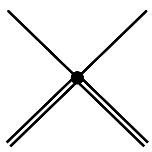
which we will denote as

$\Delta_L =$ (2.43)

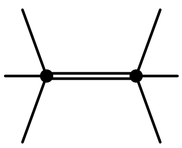
$\Delta_H =$ (2.44)

The Feynman rules for the interactions are

 $= -\lambda$  $= -\lambda$

 $= -\lambda$  $= -\lambda$

At tree level, we have diagrams such as

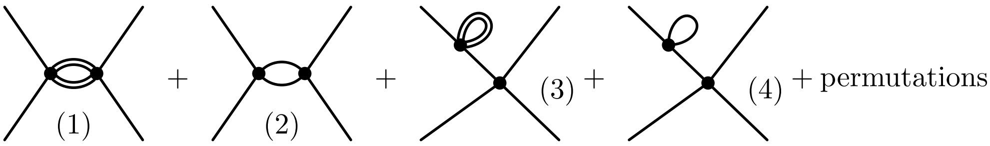
 $= \dfrac{\lambda^2}{(p_1+p_2+p_3)^2+m^2}\Theta(|p_1+p_2+p_3| > \Lambda b),$ (2.45)

[1] In Minkowski space

$$S_M = \int d^dx_M \left[\frac{1}{2}(\partial_\mu^M\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right] = i\int d^dx_E \left[\frac{1}{2}(\partial_\mu^E\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right] = iS_E$$

which corresponds to a $\frac{1}{(\Lambda b)^2}$-suppressed ϕ_L^6 interaction after integrating out ϕ_H.

However, we are mainly interested in the behavior of the ϕ^4 interaction. In this case, there is no tree-level contribution, but one-loop diagrams of the form



(2) + (4) are diagrams in the low-energy theory, so we only need to evaluate (1) and (3). It turns out that (3) only contributes to the mass term ϕ_L^2, but not to the interaction. This leaves diagram (1), for which we get

$$D_1 = \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \frac{1}{(k + p_1 + p_2)^2 + m^2} \Theta(|k| > \Lambda b) \Theta(|k| < \Lambda)$$
$$\times \Theta(|k + p_1 + p_2| > \Lambda b) \Theta(|k + p_1 + p_2| < \Lambda). \tag{2.46}$$

Since $m \ll \Lambda b$, $p_i^\mu \ll \Lambda b$ (the *external* momenta can be taken arbitrarily small), we can Taylor expand on the level of the integrand. Higher orders in the expansion are suppressed by $\frac{m}{\Lambda}$, $\frac{p^\mu}{\Lambda}$ and match onto irrelevant operators. This gives

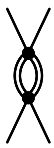
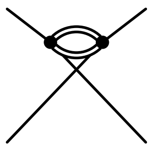
$$D_1 = \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2} \Theta(|k| > \Lambda b) \Theta(|k| < \Lambda) + \dots \tag{2.47}$$
$$= \frac{\lambda^2}{2} \frac{\Omega_d}{(2\pi)^d} \int_{b\Lambda}^{\Lambda} dk\, k^{d-5} \tag{2.48}$$
$$= \frac{\lambda^2}{2} \frac{\Omega_d}{(2\pi)^d} \frac{\Lambda^{d-4} - (\Lambda b)^{d-4}}{d-4}, \tag{2.49}$$

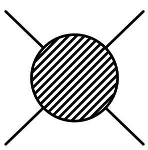
with $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ and

$$\lim_{d \to 4} \frac{\Lambda^{d-4} - (\Lambda b)^{d-4}}{d-4} = \lim_{d \to 4} \frac{1 - b^{d-4}}{d-4} = -\log b. \tag{2.50}$$

Further, for $d = 4$ we have $\Omega_4 = 2\pi^2$ and thus

$$D_1 = \frac{\lambda^2}{16\pi^2} \log \frac{1}{b}. \tag{2.51}$$

Accounting also for the crossed diagrams  and  yields the result



$$= \frac{3\lambda^2}{16\pi^2} \log \frac{1}{b} + \mathcal{O}(\lambda^4). \tag{2.52}$$

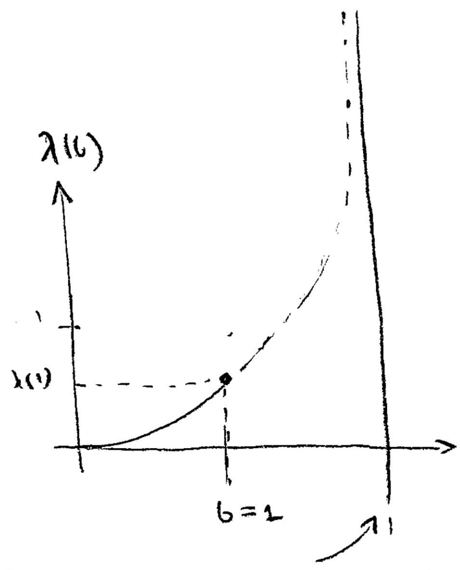


Figure 2.2.: The curved arrow denotes the Landau pole at $-1 = \frac{3}{16\pi^2}\lambda(1)\log\frac{1}{b}$.

In the low-energy theory, this contribution must arise from $-\int d^4x \frac{\lambda'}{4!}\phi_L^4$, so we can identify

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\log\frac{1}{b} \tag{2.53}$$

The coupling gets *weaker* when high-energy modes are integrated out! Let us imagine that we integrate the high-energy physics little by little, so integrating

$$d\lambda = +\frac{3\lambda^2}{16\pi^2} d\log b \tag{2.54}$$

we find

$$\int_{\lambda(1)}^{\lambda(b)} \frac{d\lambda}{\lambda^2} = \frac{-1}{\lambda(b)} + \frac{1}{\lambda(1)} = \frac{3}{16\pi^2}\log b, \tag{2.55}$$

and thus

$$\lambda(b) = \frac{\lambda(1)}{1 + \frac{3}{16\pi^2}\lambda(1)\log\frac{1}{b}}. \tag{2.56}$$

Our analysis was a bit simplistic, in that we only looked at one operator, ϕ^4, and did not include the effects of irrelevant operators. Also, our conclusion is only valid at small coupling. Nevertheless, all available (perturbative and non-perturbative) evidence suggests that the behavior persists at arbitrary values of λ. Since the coupling becomes small, one needs to start out with a sufficiently strong coupling at large values of the cutoff

$$\lambda(1) = \frac{\lambda(b)}{1 - \frac{3}{16\pi^2}\lambda(b)\log\frac{1}{b}}. \tag{2.57}$$

Since $\lambda(1) \to \infty$ for $\log\frac{1}{b} = \frac{16\pi^2}{3}\frac{1}{\lambda(b)}$ it appears that the cutoff cannot be arbitrarily large, $\log\frac{1}{b} = \log\frac{\Lambda}{E} \le \frac{16\pi^2}{3}\frac{1}{\lambda(b)}$.

Since we did our analysis in perturbation theory, the extrapolation $\lambda \to \infty$ is obviously not very meaningful. Theories that have the property that the cutoff cannot be chosen arbitrarily large for non-vanishing λ are called trivial. All evidence strongly suggests that $\lambda\phi^4$ is a trivial theory.

3. Continuum effective theory

The construction of the Wilsonian effective action is physically very intuitive, and leads to a new perspective on renormalization. However, actually integrating out the physics above a cutoff Λ is often as difficult as solving the theory. Instead of integrating out the high-energy physics, it is much simpler to work without a hard cutoff and to treat the effective theory like a standard continuum field theory. To get the effective theory, one follows a number of steps, which we now discuss in turn:

1. Identify the low-energy degrees of freedom. This can be simple: e.g., if we consider a theory with a very heavy particle and weak coupling, the low-energy degrees of freedom are simply all light particles. In other cases it is not trivial: in QCD, the low-energy degrees of freedom are pions, kaons, protons, neutrons, etc. and not the quarks and gluons in the high-energy Lagrangian.

2. Construct the most general low-energy $\mathcal{L}_{\text{eff}}$ consistent with the symmetries of the full theory. Order the operators in $\mathcal{L}_{\text{eff}}$ by their dimension.

3. Matching. To determine the coupling constants in $\mathcal{L}_{\text{eff}}$ calculate a number of correlation functions (or scattering amplitudes) in both the full and the effective theory. Expand the full theory result around the low-energy limit and adjust the Wilson coefficients in $\mathcal{L}_{\text{eff}}$ in such a way that the full and EFT results agree.

4. RG improvement. The perturbative expansion of the Wilson coefficients can be improved by using RG equations for the coefficients.

It is simplest to use dimensional regularization (and the $\overline{\text{MS}}$ scheme) in both the full and the effective theory. At first sight it seems troubling to work without a hard cutoff and integrate out to arbitrarily high energies even in the low-energy theory, which is not valid at high energies. However, we know from Wilson that we can absorb arbitrary high-energy physics into the couplings of $\mathcal{L}_{\text{eff}}$. By adjusting the couplings, we can thus obtain the correct low-energy results despite the incorrect behavior of our amplitudes at high energies.

Let us use a toy model with a heavy and light scalar field to illustrate the above steps. Our full theory is

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial_\mu \phi_L)^2 - \frac{m^2}{2}\phi_L^2 + \frac{1}{2}(\partial_\mu \phi_H)^2 - \frac{M^2}{2}\phi_H^2 \\ &- \frac{\lambda_L}{4!}\phi_L^4 - \frac{\lambda_H}{4!}\phi_H^4 - \frac{\lambda_{HL}}{4}\phi_L^2\phi_H^2 - \frac{g}{2}\phi_H\phi_L^2.\end{aligned}$$

Note that $\mathcal{L}$ is symmetric under $\phi_L \to -\phi_L$. To renormalize the theory we need to include also

$$\delta\mathcal{L} = A + B\phi_H + C\phi_H^3, \tag{3.1}$$

but we assume that A, B, C are renormalized to zero (they are not relevant for the discussion). Now let us follow the different steps to construct $\mathcal{L}_{\text{eff}}$:

1. Low-energy degrees of freedom: ϕ_L, for energies $E \ll M$.

2. Effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}} =& \frac{1}{2}(\partial_\mu \phi_L)^2 - \frac{\tilde{m}^2}{2}\phi_L^2 - \frac{\tilde{\lambda}}{4!}\phi_L^4 - \frac{1}{2}\frac{C_{2,4}}{M^2}\phi_L \Box^2 \phi_L \\ &- \frac{1}{6!}\frac{C_{6,0}}{M^2}\phi_L^6 - \frac{1}{4!}\frac{C_{4,2}}{M^2}\phi_L^2 \Box \phi_L^2 + \mathcal{O}\Big(\frac{1}{M^4}\Big).\end{aligned}$$

   To show that all other $d = 6$ operators reduce to these three one uses integration by parts and drops total derivatives.

3. Matching. To extract the values of $\tilde{m}$, $\tilde{\lambda}$, $C_{6,0}$, and $C_{4,2}$ we will now calculate the two-, four-, and six-point functions. The effect of $C_{2,4}$ can be removed via a field redefinition, see Sec. 3.2.

3.1. Tree-level matching

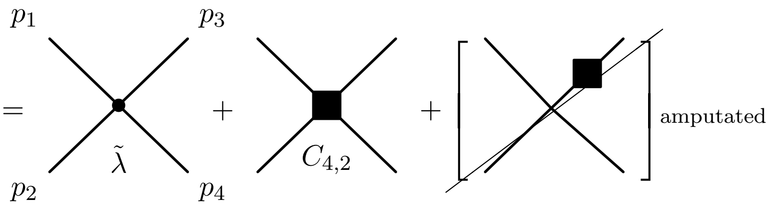
We denote the amplitude for the n-point function by $\mathcal{M}_n = -i\Gamma_n$.

Effective theory computation

The two-point function is just the inverse of the propagator

$$i\Gamma_2 = iG^{-1}(p) = (p^2 - \tilde{m}^2). \tag{3.2}$$

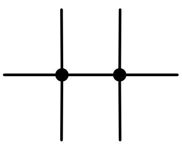
The four-point function corresponds to the scattering of four like particles, leading to

$i\Gamma_4 =$ 

(3.3)

$$= \tilde{\lambda} - \frac{C_{4,2}}{M^2}\frac{1}{3}\left[(p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2\right], \tag{3.4}$$

where the result for the diagram with $C_{4,2}$ is derived in Appendix B. Finally, the six-point function involves a contact term proportional to $C_{6,0}$ and diagrams composed of two insertions of $\tilde{\lambda}$, which we will not need to consider:

$i\Gamma_6 = \frac{C_{6,0}}{M^2} +$  $+ \dots$ (3.5)

Full theory computation

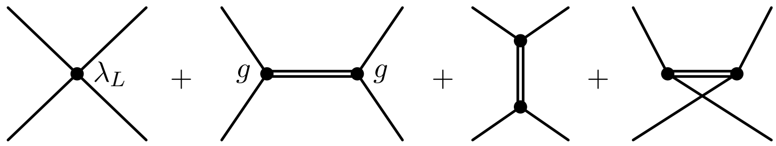
The propagator in the full theory is

$$i\Gamma_2 = iG^{-1}(p) = (p^2 - m^2), \tag{3.6}$$

so that

$$\tilde{m} = m + \mathcal{O}(\lambda), \qquad C_{2,4} = \mathcal{O}(\lambda), \tag{3.7}$$

where the second condition arises because $C_{2,4}$ only leads to a p^4 term in the two-point function. The four-point function is given by

$i\Gamma_4 =$  (3.8)

$$= \lambda_L + i(-ig)^2 \left[\frac{i}{(p_1+p_2)^2 - M^2} + \frac{i}{(p_1-p_3)^2 - M^2} + \frac{i}{(p_1-p_4)^2 - M^2} \right]. \tag{3.9}$$

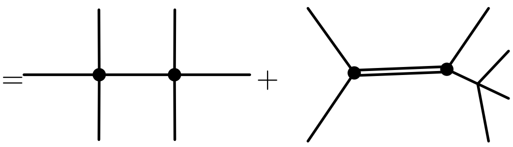
Since $p_i \ll M$ at low energies, we can Taylor expand Γ_4, leading to

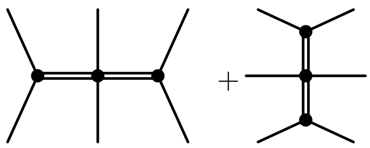
$$i\Gamma_4 = \lambda_L - \frac{3g^2}{M^2} - \frac{g^2}{M^4}\left[(p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2\right]. \tag{3.10}$$

Comparison with the EFT result gives our first non-trivial matching condition

$$\tilde{\lambda} = \lambda_L - \frac{3g^2}{M^2}, \qquad C_{4,2} = \frac{3g^2}{M^2}. \tag{3.11}$$

Finally, we have the matching of the six-point function, involving the diagrams

$i\Gamma_6 =$  $+ \dots$ (3.12)

$+$  $+ \dots$ (3.13)

The diagrams in the first line are one-particle-reducible with respect to the light field ϕ_L. These diagrams are automatically reproduced since we matched the four-point function. Only the diagrams in the second line will contribute to the matching on C_6. Since the operator does not involve derivatives it is sufficient to compute Γ_6 for vanishing momenta ($p_i = 0$). The resulting matching condition is

$$\frac{C_{6,0}}{M^2} = i(-ig)^2(-i\lambda_{HL})\left(\frac{i}{-M^2}\right)^2 \times 90, \tag{3.14}$$

$$C_{6,0} = 90\lambda_{HL}\frac{g^2}{M^2}. \tag{3.15}$$

The combinatorial factor is given as $90 = 6! \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2}$, because two permutations each at the $\phi_L^2\phi_H$ and $\phi_L^2\phi_H^2$ vertices are identical. This completes the construction of the effective theory at tree level.

3.2. Field redefinitions

With our matching computation we ensured that our effective theory reproduces the full theory result for the off-shell Green's functions. However, if we are only interested in physical quantities, such as scattering amplitudes, we can simplify the Lagrangian using field redefinitions.

As an example, consider

$$\phi_L \to \left[1 + \frac{\alpha}{M^2}\Box\right]\phi_L. \tag{3.16}$$

Inserting this into $\mathcal{L}_{\text{eff}}$ and neglecting $\frac{1}{M^4}$ terms, we get

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}_{\text{eff}} - \frac{\alpha}{M^2}\phi_L\Big(\Box + m^2 + \frac{\tilde{\lambda}}{3!}\phi_L^2\Big)\Box\phi_L, \tag{3.17}$$

where we used an integration by parts for the kinetic term. By choosing $\alpha = -\frac{1}{2}C_{2,4}$, we can cancel the term $-\frac{1}{2}\frac{C_{2,4}}{M^2}\phi\Box^2\phi$ in $\mathcal{L}_{\text{eff}}$ so that $\mathcal{L}'_{\text{eff}}$ no longer contains this term:

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}_{\text{eff}}|_{C_{2,4}\to 0} + \frac{C_{2,4}}{2M^2}\phi_L(m^2 + \frac{\tilde{\lambda}}{3!}\phi_L^2)\Box\phi_L. \tag{3.18}$$

Note that the effect of the field redefinition can be obtained by using the leading-order equation of motion (EOM)

$$\Big(\Box + m^2 + \frac{\tilde{\lambda}}{3!}\phi_L^2\Big)\phi_L = 0 \tag{3.19}$$

in order to eliminate higher-power terms in the Lagrangian.

In general, using redefinitions

$$\phi \to \phi + \Big(\frac{1}{M^2}\Big)^n f(\phi) = \phi + \delta\phi \tag{3.20}$$

generates

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}_{\text{eff}} + \underbrace{\Big(\frac{1}{M^2}\Big)^n f(\phi)}_{\delta\phi}\underbrace{\left[\Box\phi + m^2\phi + \frac{\tilde{\lambda}}{3!}\phi^3\right]}_{\text{EOM, from }\frac{\delta S}{\delta\phi}} + \mathcal{O}\left(\Big(\frac{1}{M^2}\Big)^{n+1}\right), \tag{3.21}$$

and allows one to systematically eliminate EOM terms from $\mathcal{L}_{\text{eff}}$.

These field redefinitions have to leave the physics content of the theory unchanged. This is true because:

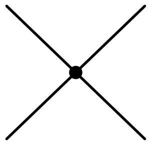
1. ϕ and ϕ' have the same quantum numbers, so after inserting states

$$\lim_{x^0\to\infty}\langle 0|\, T\{\phi(x_1)\ldots\}\,|0\rangle = \sum_x \langle 0|\,\phi(x_1)\,|x\rangle\,\langle x|\,T\{\ldots\}\,|0\rangle \tag{3.22}$$

   the same amplitudes can be extracted from the theory, only the Z-factors $\langle 0|\,\phi(0)\,|x\rangle = Z^{1/2}$ change.

2. The Jacobian $\det\left(\frac{\delta\phi}{\delta\phi'}\right)$ is trivial, at least in dimensional regularization (see below).

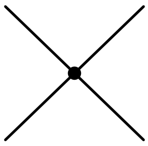
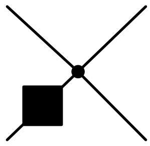
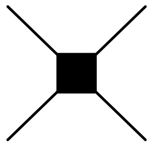
Let us illustrate the first point using an example. The tree-level $2 \to 2$ scattering amplitude in ϕ^4 theory is

$$\mathcal{M} =$$

$$= -\tilde{\lambda} \tag{3.23}$$

Let us now calculate this amplitude after the field redefinition $\phi \to \big(1 + \frac{\alpha}{M^2}\Box\big)\phi$.

$$\mathcal{L}' = \mathcal{L} - \frac{\alpha}{M^2}\phi\Big(\Box + m^2 + \frac{\tilde{\lambda}}{3!}\phi^2\Big)\Box\phi, \tag{3.24}$$

which gives

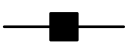
$$\mathcal{M} =$$

$$+$$

$$+ \dots +$$

(3.25)

external leg correction,
removed by amputation

$$= -\tilde{\lambda}\left(1 - \frac{\alpha}{M^2}\frac{4!}{3!} \times \frac{1}{4}\underbrace{\sum_{i=1}^{4} p_i^2}_{4m^2}\right)\left(\sqrt{Z}\right)^4 \tag{3.26}$$

$$= -\tilde{\lambda}\left(1 - \frac{\alpha}{M^2}4m^2\right)\left(\sqrt{Z}\right)^4. \tag{3.27}$$

To get the Z-factor, we need to evaluate the two-point function:

$$+$$

$$= \frac{i}{p^2 - m^2} - i\frac{2\alpha}{M^2}\frac{i}{p^2 - m^2}(-p^2)(-p^2 + m^2)\frac{i}{p^2 - m^2} \tag{3.28}$$

$$= \frac{i}{p^2 - m^2}\Big(1 + \frac{2\alpha}{M^2}p^2\Big) = \frac{i(1 + \frac{2\alpha}{M^2}m^2)}{p^2 - m^2} + \text{"non-pole"}, \tag{3.29}$$

so that the wave-function renormalization becomes

$$Z = 1 + \frac{2\alpha}{M^2}m^2. \tag{3.30}$$

For the amplitude, we thus have

$$\mathcal{M} = -\tilde{\lambda}\Big(1 - \frac{\alpha}{M^2}4m^2\Big)\Big(1 + \frac{2\alpha}{M^2}m^2\Big)^2 = -\tilde{\lambda} + \mathcal{O}\Big(\frac{1}{M^4}\Big), \tag{3.31}$$

precisely as before the field redefinition (at the order considered).

Finally, let us show why the Jacobian is trivial:

$$\int \mathcal{D}\phi = \int \mathcal{D}\phi' \det\left(\frac{\delta\phi}{\delta\phi'}\right) \tag{3.32}$$

In our case, $\phi \to \phi' + \left(\frac{1}{M^2}\right)^n f(\phi')$,

$$\frac{\delta\phi}{\delta\phi'} = \delta(x - x') + \left(\frac{1}{M^2}\right)^n f'(\phi'(x))\delta(x - x'). \tag{3.33}$$

The Jacobian can be written as

$$\det\left(\frac{\delta\phi}{\delta\phi'}\right) = \int \mathcal{D}c \int \mathcal{D}\bar{c}\, \exp\left[i \int d^dx \int d^dy\, \bar{c}(x) \frac{\delta\phi}{\delta\phi'} c(y)\right] \tag{3.34}$$

$$= \int \mathcal{D}c \int \mathcal{D}\bar{c}\, \exp\left[i \int d^dx\, \bar{c}(x) \left[1 + \left(\frac{1}{M^2}\right)^n f'(\phi')\right] c(x)\right], \tag{3.35}$$

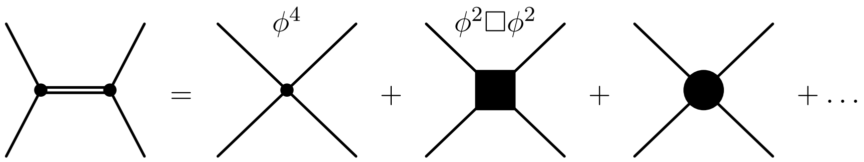
where c, $\bar{c}$ are Grassmann fields.

Since the term involving f' is suppressed by $\frac{1}{M^2}$, it can be treated as a perturbation. The corresponding "ghost" diagrams are loops of a "fermion" with "propagator" $\frac{i}{1}$ (i.e., the ghost fields do not propagate since they do not have a kinetic term). Such loops that do not involve a scale, e.g., $\int d^dk \left(\frac{i}{1}\right)^n$, vanish in dimensional regularization, hence $\det(\frac{\delta\phi}{\delta\phi'}) = 1$.

3.3. Matching at higher orders

The general method to perform the matching is always the same: compute the same quantity both in the EFT and in the full theory and adjust the Wilson coefficients ("coupling constants") in the EFT in such a way that the results agree to the given order of the low-energy expansion.

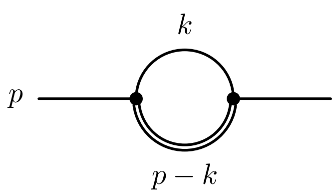
At the tree level, it is simple to understand that the procedure works: the heavy particles are always far off-shell and their propagators get expanded into a polynomial. Since $\mathcal{L}_{\text{eff}}$ contains all higher-dimensional operators, it generates the most general polynomial contribution to the amplitude and by adjusting the couplings, we reproduce the full theory result. Schematically:



(3.36)

$$\frac{1}{p^2 - M^2} = -\frac{1}{M^2} - \frac{1}{M^2}\frac{p^2}{M^2} - \dots \tag{3.37}$$

At higher orders, there are several interesting new features that emerge from the loop diagrams. Let us consider the self-energy diagram in the full theory



The following complications occur:

1. The loop momentum k can be small or large: the heavy propagator is *not* always off shell.

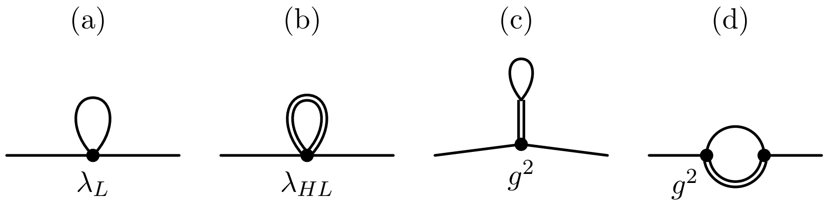
2. Renormalization: there are UV divergences both in the full and effective theory, and we thus have to renormalize all couplings/Wilson coefficients.

3. Loop diagrams are non-trivial (i.e., non-analytic) functions of external momenta and masses, but the Wilson coefficients C_i can only depend on the high-energy scale M ($\mathcal{L}_{\text{eff}}$ must be local). Accordingly, the non-trivial parts of the amplitudes must be present in the EFT loop computation, as they cannot be obtained from the C_i. In the end, this has to work out because the low-energy dynamics of $\mathcal{L}_{\text{full}}$ and $\mathcal{L}_{\text{eff}}$ are the same, but it would be nice to see this in an example.

4. Due to renormalization in $\mathcal{L}_{\text{eff}}$, the Wilson coefficients depend on the renormalization scale μ: $C_i = C_i(\mu)$. The coefficients involve terms $g^2 \log \frac{\mu}{M}$, $\lambda_{HL} \log \frac{\mu}{M}$ etc.

To see how all this works in practice, we will now perform a one-loop matching computation in the simplest possible setting: we will compute the two-point function and the one-loop corrections to $\tilde{m}$, the mass of the light particle in $\mathcal{L}_{\text{eff}}$. The relevant diagrams are:

Full theory :



EFT :

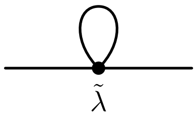


Diagram (a): Writing $i\Sigma = -\mathcal{M}$ for the two-point functions, the first contribution is given by

$$i\Sigma^{(a)} = i(-i\lambda_L)\frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\frac{i}{k^2 - m^2}\mu^{2\epsilon}, \tag{3.38}$$

where the factor one half is the symmetry factor and $\mu^{2\epsilon}$ is the renormalization scale, introduced to make λ dimensionless in $d = 4 - 2\epsilon$. Using the results from Appendix A we find

$$\begin{aligned} i\Sigma^{(a)} &= \frac{\lambda_L}{2}(4\pi)^{-\frac{d}{2}}\Gamma\Big(1 - \frac{d}{2}\Big)(m^2)^{\frac{d}{2}-1}\mu^{2\epsilon} \\ &= \frac{m^2\lambda_L}{32\pi^2}\left[-\frac{1}{\epsilon} + \gamma_E - \log(4\pi) - 1 + \log\frac{m^2}{\mu^2} + \mathcal{O}(\epsilon)\right]. \end{aligned} \tag{3.39}$$

Since $-\gamma_E + \log(4\pi)$ always appear together with the $\frac{1}{\epsilon}$ term, and since they are numerically not small, it is customary to remove not only the $\frac{1}{\epsilon}$ terms (minimal subtraction

$\equiv$ MS scheme), but the entire combination ($\equiv$ $\overline{\text{MS}}$ scheme) by renormalization. One way to achieve this, is to set $\mu^2 = \bar{\mu}^2 e^{\gamma_E}/(4\pi)$ so that

$$i\Sigma^{(a)} = \frac{m^2\lambda_L}{32\pi^2}\left[-\frac{1}{\epsilon} - 1 + \log\frac{m^2}{\bar{\mu}^2} + \mathcal{O}(\epsilon)\right]. \tag{3.40}$$

Diagram (b): The calculation is analogous to diagram (a), leading to

$$i\Sigma^{(b)} = \frac{M^2\lambda_{HL}}{32\pi^2}\left[-\frac{1}{\epsilon} - 1 + \log\frac{M^2}{\bar{\mu}^2} + \mathcal{O}(\epsilon)\right]. \tag{3.41}$$

Diagram (c): The expression is the same as diagram (a) apart from the additional heavy propagator

$$i\Sigma^{(c)} = i\frac{1}{2}(-ig)^2\frac{i}{-M^2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2}\mu^{2\epsilon} \tag{3.42}$$

$$= \frac{g^2}{32\pi^2}\frac{m^2}{M^2}\left[\frac{1}{\epsilon} + 1 - \log\frac{m^2}{\bar{\mu}^2} + \mathcal{O}(\epsilon)\right]. \tag{3.43}$$

Diagram (d): This is the only non-trivial contribution, namely

$$i\Sigma^{(d)} = i(ig)^2\int\frac{d^dk}{(2\pi)^d}\frac{i^2\mu^{2\epsilon}}{[(k+p)^2-m^2](k^2-M^2)}. \tag{3.44}$$

We follow the strategy described in Appendix A to bring the integral into standard form, by introducing a Feynman parameter

$$\frac{1}{[(k+p)^2-m^2](k^2-M^2)} = \int_0^1\frac{dx}{[(k+px)^2 + \underbrace{x(1-x)p^2 - xm^2 - (1-x)M^2}_{\equiv -\Delta(x)}]^2} \tag{3.45}$$

and shifting the integration variable by $k \to k - px$. This yields

$$i\Sigma^{(d)} = -\frac{g^2\mu^{2\epsilon}}{(4\pi)^{d/2}}\Gamma(\epsilon)\int_0^1 dx[\Delta(x)]^{-\epsilon} \tag{3.46}$$

$$= \frac{g^2}{16\pi^2}\left[-\frac{1}{\epsilon} + \log\frac{M^2}{\bar{\mu}^2} + \int_0^1 dx\log\frac{xm^2+(1-x)M^2-x(1-x)p^2}{M^2}\right] \tag{3.47}$$

$$= \frac{g^2}{16\pi^2}\left[-\frac{1}{\epsilon} + \log\frac{M^2}{\bar{\mu}^2} - 1 - \frac{p^2}{2M^2} - \frac{m^2}{M^2}\log\frac{m^2}{M^2} + \mathcal{O}\Big(\frac{1}{M^4}\Big)\right], \tag{3.48}$$

where in the last step we expanded in $1/M^2$. To this end, it is easiest to first expand in p^2, which reduces the integral to

$$\int_0^1 dx\frac{-x(1-x)p^2}{xm^2+(1-x)M^2} + \int_0^1 dx\log\frac{xm^2+(1-x)M^2}{M^2} \tag{3.49}$$

$$= -\frac{p^2}{2M^2} - 1 + \frac{m^2\log\frac{m^2}{M^2}}{m^2-M^2} + \mathcal{O}\Big(\frac{1}{M^4}\Big). \tag{3.50}$$

EFT diagram: Finally, the effective theory diagram reads

$$i\Sigma = \frac{m^2\tilde{\lambda}}{32\pi^2}\left[-\frac{1}{\epsilon} - 1 + \log\frac{m^2}{\bar{\mu}^2}\right]. \tag{3.51}$$

We now renormalize, i.e., absorb the $\frac{1}{\epsilon}$-pieces of Σ into the couplings of the full and effective theory and then calculate the difference

$$\Delta = i\Sigma_{\text{full}} - i\Sigma_{\text{eff}}, \tag{3.52}$$

to be absorbed into the couplings of $\mathcal{L}_{\text{eff}}$. Since we will end up also with a wave-function renormalization, we write the effective theory Lagrangian as

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}Z(\partial_\mu\phi)^2 - \frac{\tilde{m}^2}{2}Z\phi^2 - \frac{\tilde{\lambda}}{4!}Z^2\phi^4, \tag{3.53}$$

with $Z = 1 + \mathcal{O}(\lambda)$ (the one-loop Z-factor in the full theory happens to be trivial). The matching condition then reads

$$0 = \Delta = m^2 - p^2 - Z\tilde{m}^2 + Zp^2 + i\Sigma_{\text{full}}^{\text{(1-loop)}} - i\Sigma_{\text{eff}}^{\text{(1-loop)}}. \tag{3.54}$$

In order for this to work, all low-energy physics has to drop out of Δ. In particular, the $\log m^2$ pieces in Σ_{full} and Σ_{eff} have to cancel! Let us look at these terms

$$\frac{32\pi^2}{m^2}\Delta = \log m^2\left[\lambda_L - \frac{3g^2}{M^2} - \tilde{\lambda}\right] + \cdots \tag{3.55}$$

In our tree-level calculation we found $\tilde{\lambda} = \lambda_L - \frac{3g^2}{M^2}$, so our $\mathcal{L}_{\text{eff}}$ indeed reproduces the low-energy part of the full theory. The remaining terms are

$$\Delta = m^2 - Z\tilde{m}^2 + \frac{1}{16\pi^2}\left[g^2\Big(1 + \frac{m^2}{M^2}\Big) + \frac{M^2}{2}\lambda_{HL}\right]\Big(\log\frac{M^2}{\bar{\mu}^2} - 1\Big) - p^2 + Zp^2 - \frac{g^2}{32\pi^2}p^2. \tag{3.56}$$

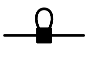
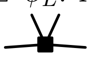
From the momentum-dependent term we can read off

$$Z = 1 + \frac{g^2}{32\pi^2}, \tag{3.57}$$

and therefore

$$\tilde{m}^2 = m^2\Big(1 - \frac{g^2}{32\pi^2}\Big) + \left[g^2\Big(1 + \frac{m^2}{M^2}\Big) + \frac{M^2}{2}\lambda_{HL}\right]\frac{\log\frac{M^2}{\bar{\mu}^2} - 1}{16\pi^2} + \cdots \tag{3.58}$$

Note the presence of the $M^2\lambda_{HL}$ contribution. The same contributions will also arise in the physical mass m_{ph} determined by $i\Sigma(p^2 = m_{\text{ph}}^2) = 0$. If m_{ph} is small, this implies a large cancellation. This is again the statement that small scalar masses are unnatural.

Expanding the full theory result to higher power in p^2, one can determine the Wilson coefficients of operators $\frac{1}{2}C_{2,2n}\phi_L\Box^n\phi_L$. Note that in the matching for $\phi_L\Box^2\phi_L$, a new effective theory diagram , where  $\propto \phi_L^2\Box\phi_L^2$, contributes.

By computing the four- and six-point functions one can then determine also $\tilde{\lambda}$, $C_{4,2}$, and $C_{6,0}$ to one-loop accuracy. In this case, the number of diagrams becomes quite large. On the other hand, for $\tilde{\lambda}$, the four-point function at vanishing momentum is sufficient.

3.4. Power counting

In the matching calculation we have assumed that the higher-order Lagrangians do not contribute at leading power. At tree level, this is obvious, since the operators in the power-suppressed Lagrangians have additional derivatives and/or fields. But in loop integrals the momenta are large, so it is less obvious that higher-derivative terms are suppressed.

For instance, the contribution of $\frac{1}{(M^2)^n}\phi_L^2\Box^n\phi_L^2$ to the two-point function at zero external momentum reads

$$\delta\Sigma \propto \frac{1}{(M^2)^n}\int d^dk\,\frac{(k^2)^n}{k^2-m^2}, \tag{3.59}$$

and the loop integral, in principle, extends over all scales. The advantage of dimensional regularization is that the loop integrals in the EFT only depend on low-energy scales, therefore by dimensional analysis $\delta\Sigma \propto (m^2)^{d/2-1}\times(m^2)^n\times\frac{1}{(M^2)^n}$, since scaleless integrals are set to zero. Therefore, the loop contribution is indeed suppressed by $\left(\frac{m^2}{M^2}\right)^n$.

Note that in a cutoff regularization

$$\frac{1}{(M^2)^n}\int^\Lambda d^dk\frac{(k^2)^n}{k^2-m^2} \propto \Lambda^{d-2}\times\frac{\Lambda^{2n}}{M^{2n}}+\cdots\,, \tag{3.60}$$

so that the loop contributions of higher-dimensional operators are not suppressed. The terms that violate the tree-level power counting are trivial cutoff terms and can thus be subtracted, but they make computations cumbersome.

In contrast, the power counting in dimensional regularization is very simple. To calculate a quantity up to $\left(\frac{1}{M^2}\right)^n$ accuracy, we need the Lagrangian up to $\left(\frac{1}{M^2}\right)^n$ and the $\left(\frac{1}{M^2}\right)^n$ corrections arise from diagrams with: a single vertex from $\mathcal{L}_{\text{eff}}^{(n)}$, or one from $\mathcal{L}_{\text{eff}}^{(n-m)}$ and one from $\mathcal{L}_{\text{eff}}^{(m)}$, or one from $\mathcal{L}_{\text{eff}}^{(n-m_1-m_2)}$ and one from $\mathcal{L}_{\text{eff}}^{(m_1)}$ and $\mathcal{L}_{\text{eff}}^{(m_2)}$ etc.

3.5. Renormalization-group improved perturbation theory

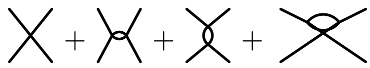
The Wilson coefficients C_i in $\mathcal{L}_{\text{eff}}$ depend on the coupling constants of the full theory as well as the large energy scale M. The dependence on M is logarithmic. Schematically, we have

$$\begin{aligned} C_i(M,\mu,\lambda) &= C_i^{(0,0)} + \lambda(\mu)\left[C_i^{(1,1)}\log\frac{M^2}{\mu^2}+C_i^{(1,0)}\right] \\ &\quad + \lambda^2(\mu)\left[C_i^{(2,2)}\log^2\frac{M^2}{\mu^2}+C_i^{(2,1)}\log\frac{M^2}{\mu^2}+C_i^{(2,0)}\right]+\cdots\,, \end{aligned}$$

where the coefficients $C_i^{(n,m)}(m\le n)$ are pure numbers, determined by the matching calculation. In our matching calculation for $\tilde{m}$ we found exactly this structure, except for the fact that our full theory had several different couplings. The form of the result makes it obvious that we should choose $\mu\approx M$, otherwise perturbation theory will not work well because the $\log\frac{M^2}{\mu^2}$ terms become large.

To understand better how to treat these logarithms, let us take a look at a computation in the EFT. For the $2 \to 2$ amplitude at leading order, we get

$$\mathcal{M} = \tag{3.61}$$



$$= -\tilde{\lambda}_0 \left[1 + \frac{3\tilde{\lambda}_0}{32\pi^2} \left(-\frac{1}{\epsilon} + \log \frac{m^2}{\mu^2} + f(s,t,u) \right) \right] \tag{3.62}$$

$$= -\tilde{\lambda}(\mu) \left[1 + \frac{3\tilde{\lambda}(\mu)}{32\pi^2} \left(\log \frac{m^2}{\mu^2} + f(s,t,u) \right) \right], \tag{3.63}$$

where $\tilde{\lambda}(\mu)$ is the $\overline{\mathrm{MS}}$ renormalized coupling and for simplicity we will write $\bar{\mu} \to \mu$ from now on. The finite part of the amplitude can be written as

$$f(s,t,u) = V(s) + V(t) + V(u), \qquad V(s) = \frac{1}{3} \int_0^1 dx \log \frac{m^2 - x(1-x)s}{m^2}, \tag{3.64}$$

with the Mandelstam variables

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2, \tag{3.65}$$

but its precise form does not matter for the following discussion.

To get reasonable higher-order corrections, we need $\mu \approx m$, which leads to conflicting conditions:

1. matching requires $\mu \approx M$,
2. EFT matrix element require $\mu \approx m$,
3. starting point was $m \ll M$.

This problem would manifest itself in the full theory as terms of the form $\lambda^n \log^n \frac{m^2}{M^2}$ and results in a breakdown of perturbation theory (in the $\overline{\mathrm{MS}}$ scheme) for $m \ll M$, even if λ is very small.

Fortunately, the RG in the EFT allows us to resum the logarithmically enhanced terms to all orders by solving the RG evolution equations

$$\frac{d\tilde{\lambda}(\mu)}{d\log\mu} = \mu \frac{d\tilde{\lambda}(\mu)}{d\mu} = \beta(\tilde{\lambda}(\mu)) \tag{3.66}$$

and

$$\frac{dC_i(\mu)}{d\log\mu} = \gamma_{ij}(\tilde{\lambda}(\mu)) C_j(\mu). \tag{3.67}$$

This second equation is a matrix equation. Operators of the same dimension "mix," i.e., their RG evolution is coupled.

The general strategy is then as shown in Fig. 3.5. Let us illustrate this procedure for the leading-order four-point function. To get the β-function, we use

$$\frac{d}{d\log\mu} \mathcal{M} = 0, \tag{3.68}$$

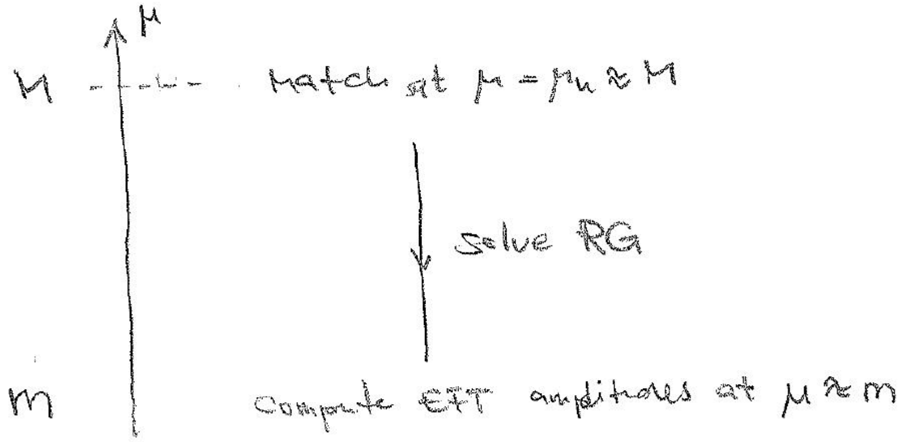


Figure 3.1.: General strategy for matching in EFT.

since the physical amplitude is μ-independent. This gives

$$\frac{d}{d\log(\mu)}\tilde{\lambda}(\mu) = \beta(\tilde{\lambda}) = \frac{3\tilde{\lambda}^2(\mu)}{16\pi^2} + \cdots\,, \tag{3.69}$$

which can be solved via a separation of variables

$$\int_{\tilde{\lambda}(\mu_n)}^{\tilde{\lambda}(\mu)} \frac{d\tilde{\lambda}}{\tilde{\lambda}^2} = \frac{3}{32\pi^2}\log\frac{\mu^2}{\mu_n^2}, \tag{3.70}$$

yielding

$$\frac{1}{\tilde{\lambda}(\mu_n)} - \frac{1}{\tilde{\lambda}(\mu)} = \frac{3}{32\pi^2}\log\frac{\mu^2}{\mu_n^2}. \tag{3.71}$$

The solution takes the form

$$\tilde{\lambda}(\mu) = \frac{\tilde{\lambda}(\mu_n)}{1 - \frac{3}{16\pi^2}\tilde{\lambda}(\mu_n)\log\frac{\mu}{\mu_n}} = -\frac{1}{\frac{3}{16\pi^2}\log\frac{\mu}{\Lambda}}, \tag{3.72}$$

where Λ is the scale at which $\lambda \to \infty$. Using (3.71), we can eliminate all logarithms in our result:

$$\mathcal{M} = -\tilde{\lambda}(\mu)\left[1 + \tilde{\lambda}(\mu)\left(\frac{1}{\tilde{\lambda}(\mu)} - \frac{1}{\tilde{\lambda}(m)}\right) + \frac{3\tilde{\lambda}(\mu)}{32\pi^2}f(s,t,u)\right] \tag{3.73}$$

$$= -\tilde{\lambda}(\mu) \underbrace{\left[2 - \frac{\tilde{\lambda}(\mu)}{\tilde{\lambda}(m)}\right]}_{\text{evolution from } \mu \text{ to } m} + \cancel{\mathcal{O}(\tilde{\lambda}^2)}, \tag{3.74}$$

with $\mu \approx M$. This is the leading-order result in RG improved perturbation theory. It resums terms of the form $\tilde{\lambda} \times \tilde{\lambda}^n \log^n \frac{m^2}{\mu^2}$. Note that the term $\frac{\tilde{\lambda}^2}{32\pi^2} f$ has been dropped: the reason is that higher orders contain terms scaling as $\tilde{\lambda} \times \tilde{\lambda}^n \log^{n-1} \frac{m^2}{\mu^2} \sim \tilde{\lambda}^2$, which are of the same size as $\tilde{\lambda}^2 f$. To include this term we should therefore resum those logarithms as well, which is achieved by solving the evolution with the two-loop β-function:

$$\frac{d\tilde{\lambda}}{d\log\mu} = \beta(\tilde{\lambda}) = \tilde{\lambda}\left[3\frac{\tilde{\lambda}}{16\pi^2} - \frac{17}{3}\left(\frac{\tilde{\lambda}}{16\pi^2}\right)^2 + \cdots\right], \tag{3.75}$$

so that

$$\frac{3}{16\pi^2}\log\frac{\mu}{\mu_n} = \frac{1}{\tilde{\lambda}(\mu_n)} - \frac{1}{\tilde{\lambda}(\mu)} + \frac{17}{9} \times \frac{1}{16\pi^2}\log\frac{\tilde{\lambda}(\mu)}{\tilde{\lambda}(\mu_n)} + \mathcal{O}(\tilde{\lambda}). \tag{3.76}$$

In the presence of large scale hierarchies large logarithms of scale ratios destroy the perturbative expansion. The use of an EFT allows one to disentangle the different scales and, by using RG evolution, resum the logarithmically enhanced contributions. To avoid large logarithms, low-energy calculations are never performed using the SM Lagrangian directly, but an effective Lagrangian obtained by "integrating out" heavy particles such as t-quarks, Higgs, $W^{\pm}$, Z^0. Having worked out the construction of an EFT in our scalar-field toy example, we are now ready for real-life applications of this technology. To finish our discussion, let us repeat the steps needed to construct the effective theory:

1. Identify the degrees of freedom at low energy.

2. Construct the most general $\mathcal{L}$ with these degrees of freedom and the symmetries of the full theory.

    a) Higher-dimensional operators in $\mathcal{L}_{\text{eff}}$ are suppressed by $(\frac{1}{M})^n$, where M is a characteristic high-energy scale. Their contribution to observables is suppressed by $(\frac{E}{M})^n$, so only a finite number of terms is needed for a given accuracy ϵ: $n \approx \frac{\log\epsilon}{\log(E/M)}$.

    b) Field redefinitions: higher-order terms in $\mathcal{L}_{\text{eff}}$ that vanish by the leading-order EOM do not contribute to physical amplitudes and can be omitted from $\mathcal{L}_{\text{eff}}$.

3. Matching: if possible, determine the Wilson coefficients of the operators in $\mathcal{L}_{\text{eff}}$ by computing a number of quantities in both the full and the effective theory. Adjust the couplings in $\mathcal{L}_{\text{eff}}$ to reproduce the full theory result. If field redefinitions have been used, only physical quantities match, otherwise arbitrary Green's functions.

4. RG improvement: compute the anomalous dimensions and solve the RG equations for the operators in $\mathcal{L}_{\text{eff}}$, i.e.,

$$C_i(\mu) = \sum_j \underbrace{U_{ij}(\mu_n, \mu)}_{\substack{\text{Evolution}\\ \text{from } \mu_n \approx M\\ \text{to } \mu \approx E\\ \text{resums logarithms}}} \underbrace{C_j(\mu_n)}_{\substack{\text{no large logarithms}\\ \text{for } \mu_n \approx M}}. \tag{3.77}$$

Note:

Since the higher orders contain terms like $\tilde{\lambda}^n \log^n \frac{m^2}{\mu^2}$ substituting $\frac{3}{16\pi^2} \log \frac{m^2}{\mu^2} = \frac{1}{\tilde{\lambda}(\mu)} - \frac{1}{\tilde{\lambda}(m)}$ only at $\mathcal{O}(\tilde{\lambda})$ does not give the full result. Instead, follow the procedure outlined on in Fig. 3.5:

1. Solve the RG for $\tilde{\lambda}$,
$$\tilde{\lambda}(\mu) = \frac{\tilde{\lambda}(\mu_n)}{1 - \frac{3}{16\pi^2}\tilde{\lambda}(\mu_n) \log \frac{\mu}{\mu_n}} \qquad \text{for} \quad \mu_n \approx M. \tag{3.78}$$

2. Choose $\mu \approx m$. For $\mu = m$, the result for $\mathcal{M}$ does not contain large logarithms.

3. For $\mu = m$:
$$\mathcal{M} = -\tilde{\lambda}(m)\left[1 + \frac{3\tilde{\lambda}(m)}{32\pi^2} f(s,t,u)\right]. \tag{3.79}$$

4. The Standard Model at low energies

In this chapter we turn to low-energy effective theories for the interactions of the Standard Model (SM). In each case, the EFT approach simplifies calculations for a given sector of the SM, and is sometimes the only viable strategy to perform analytic calculations. We will first treat electromagnetism (Euler–Heisenberg theory), before turning to weak and strong interactions.

4.1. Euler–Heisenberg theory

Let us consider the QED Lagrangian

$$\mathcal{L}_{\mathrm{QED}}[A_\mu, \psi] = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \tag{4.1}$$

where

$$iD_\mu = i\partial_\mu - eA_\mu, \tag{4.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{e}[iD_\mu, iD_\nu]. \tag{4.3}$$

A_μ is the electromagnetic potential and ψ the electron field. Note that $\mathcal{L}_{\mathrm{QED}}$ is the most general renormalizable Lagrangian for an electron interacting with the photon field. In the SM, there are many other heavier charged particles (quarks, $W^\pm$, μ, τ), but according to EFT logic, the contributions of all the other fields only appear via $\frac{1}{M}$ suppressed operators (where $M = m_\tau, M_\pi, \dots$). In other words, $\mathcal{L}_{\mathrm{QED}}$ is the leading-order effective Lagrangian describing the interactions of $e^\pm$ and γ, and it will be appropriate as long as $E \ll 100\,\mathrm{MeV} \approx m_\mu \sim M_\pi$.

Many practical applications only involve photons at even smaller energies $E \ll 2m_e$. In this case electron–positron pairs appear only as virtual corrections and we can integrate them out, i.e., construct an effective theory involving only photons. To do so, we first encounter an interesting complication: since electron number is conserved, a given state, say with $5e^-$, will be there even for $E \to 0$. To describe such a situation correctly, one has to use non-relativistic EFT, to which we will turn in a later chapter. For now, we concentrate on the sector with zero net electrons, in which $e^\pm$ only appear as virtual particles. To do so, we can just describe the electrons as an external current and add a term

$$\mathcal{L}_J = -eA_\mu J^\mu \tag{4.4}$$

to the Lagrangian. This description should work for macroscopic charged objects, as long as we do not excite higher energy levels in their interaction with the photon. Note that this interaction is only consistent if $\partial_\mu J^\mu = 0$. Under gauge transformations

$$A^\mu \to A^\mu + \partial^\mu \phi, \tag{4.5}$$

so

$$\int d^4x\, A_\mu J^\mu \to \int d^4x A_\mu J^\mu - \int d^4x \phi \underbrace{\partial_\mu J^\mu}_{=0}. \tag{4.6}$$

Examples of configurations fulfilling $\partial_\mu J^\mu = 0$ are

1. static charge distribution: $J^\mu = (\rho(r), \mathbf{0})$
2. static current: $J^\mu = (0, \mathbf{j}(r))$ with $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$

If we now consider low-energy photons in the background of a source J^μ, we should be able to describe their interactions with the effective Lagrangian

$$\mathcal{L}_{\text{eff}}[A_\mu, J_\mu] = \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \mathcal{L}^{(8)} + \dots \tag{4.7}$$

The leading-order Lagrangian is

$$\mathcal{L}^{(4)} = -\frac{Z}{4} F^{\mu\nu} F_{\mu\nu} - e A_\mu J^\mu \tag{4.8}$$

and describes free photons. Let us now construct the operators of dimension 6 and 8, whose effects are suppressed by $\mathcal{O}(m_e^{-2})$ and $\mathcal{O}(m_e^{-4})$, respectively. To obtain $\mathcal{L}_{\text{eff}}$, one writes down all possible terms of given dimension. The number of terms can be reduced to a minimal set using

(i) Symmetries, e.g., charge conjugation. QED is invariant under

$$e \to -e, \qquad A_\mu \to -A_\mu, \qquad F_{\mu\nu} \to -F_{\mu\nu},$$

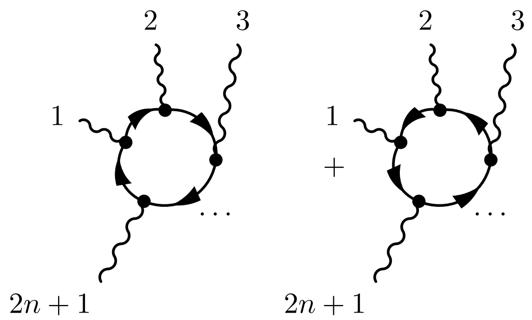
and so has to be the effective theory.

(ii) Properties of $F_{\mu\nu}$, e.g., its antisymmetry $F^{\mu\nu} = -F^{\nu\mu}$ and the Bianchi identity

$$\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0.$$

(iii) The leading-order EOM $\partial_\mu F^{\mu\nu} = J^\nu$.

We start with the $d = 6$ terms. Because of charge conjugation symmetry, $\mathcal{L}_{\text{eff}}$ must be even in $F^{\mu\nu}$. This is the EFT equivalent of Furry's theorem, which states that amplitudes with an odd number of photons vanish in QED.



$$= 0 \tag{4.9}$$

This leaves us with terms of the form $\partial^2 F^2$. Using integration by parts, we can always achieve that derivatives are not contracted with the field strength on which they act. This leaves two possible terms:

$$O_1 = F^{\mu\nu}\Box F_{\mu\nu}, \tag{4.10}$$
$$O_2 = (\partial^\rho F^{\mu\nu})(\partial_\mu F_{\rho\nu}). \tag{4.11}$$

Using the Bianchi identity on O_2, we find

$$\begin{aligned} O_2 &= (\partial^\rho F^{\mu\nu})\left[-\partial_\rho F_{\nu\mu} - \partial_\nu F_{\mu\rho}\right] && (4.12)\\ &= F^{\mu\nu}\Box F_{\nu\mu} - \partial^\rho F^{\mu\nu}\partial_\nu F_{\mu\rho} \ + \text{ total derivative} && (4.13)\\ &= -F^{\mu\nu}\Box F_{\mu\nu} - \partial^\rho F^{\nu\mu}\partial_\nu F_{\rho\mu} \ + \text{ total derivative} && (4.14)\\ &= -O_1 - O_2 \ + \text{ total derivative}, && (4.15) \end{aligned}$$

so that the terms O_1 and O_2 are equivalent, $2O_2 \hat{=} -O_1$. In addition, we can write down terms

$$O_3 = J_\mu J^\mu, \qquad O_4 = \partial_\mu F^{\mu\nu} J_\nu, \tag{4.16}$$

since J^μ has dimension $d = 3$. These two terms are equivalent upon using the EOM $\partial_\mu F^{\mu\nu} = J^\nu$. Moreover, up to total derivatives also O_2 can be brought into the form $O_2 \hat{=} \partial_\mu F^{\mu\nu}\partial^\rho F_{\rho\nu}$, in such a way that all operators become equivalent to O_3. Our final result can therefore be expressed as

$$\mathcal{L}^{(6)} = \frac{C_0}{m_e^2} J^\mu J_\mu, \tag{4.17}$$

which corresponds to a contact interaction between the source and is irrelevant for photon propagation or scattering.

The first terms involving photons appear for $d = 8$

$$\mathcal{L}^{(8)} = \frac{C_1}{m_e^4}\big(F^{\mu\nu}F_{\mu\nu}\big)^2 + \frac{C_2}{m_e^4} F_{\mu\nu}F^{\nu\sigma}F_{\sigma\rho}F^{\rho\mu}. \tag{4.18}$$

In four space-time dimensions, we can rewrite

$$F_{\mu\nu}F^{\nu\sigma}F_{\sigma\rho}F^{\rho\mu} = \frac{1}{4}(F^{\mu\nu}\tilde{F}_{\mu\nu})^2 + \frac{1}{2}\left(F^{\mu\nu}F_{\mu\nu}\right)^2, \tag{4.19}$$

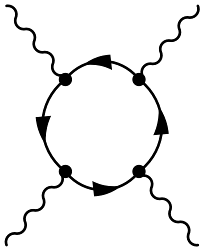
where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. However, since $\epsilon^{\mu\nu\rho\sigma}$ is only defined in $d = 4$, it is preferable not to use this relation.[1] Expressed in terms of $\mathbf{E}$ and $\mathbf{B}$ the two structures are

$$F^{\mu\nu}F_{\mu\nu} = -2\big(\mathbf{E}^2 - \mathbf{B}^2\big), \qquad \tilde{F}^{\mu\nu}F_{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}. \tag{4.20}$$

The two terms in $\mathcal{L}^{(8)}$ describe four-point interactions, but since they are suppressed by $\mathcal{O}(m_e^{-4})$, they will be very weak at low energies where the EFT applies. In QED, these interactions arise from fermion loops

[1] To derive it, use:

$$\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\nu_1\nu_2\nu_3\nu_4} = -\begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \dots \\ \vdots & & \\ \delta^{\nu_1}_{\mu_4} & \dots & \delta^{\nu_4}_{\mu_4} \end{vmatrix}$$



Before considering these diagrams, let us study the low-energy $\gamma\gamma \to \gamma\gamma$ scattering cross section. This cross section scales according to

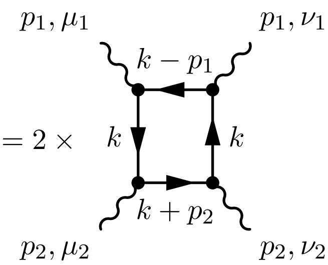
$$d\sigma \sim \left(\frac{1}{m_e^4}\right)^2 E^6 e^8, \qquad e^8 \propto \alpha^4. \tag{4.21}$$

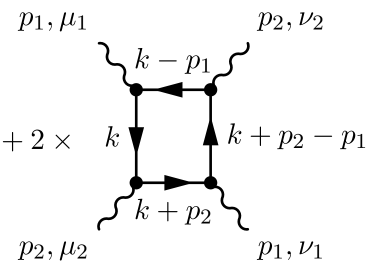
The factor E^6 is required to get the correct dimension $d\sigma \sim E^{-2}$. An explicit computation yields the unpolarized cross section:

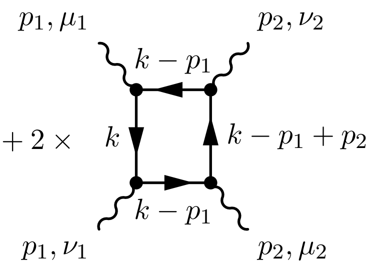
$$\frac{d\sigma_{\gamma\gamma}}{d\Omega} = \frac{1}{4\pi^2}(48C_1^2 + 40C_1C_2 + 11C_2^2)\frac{E^6}{m_e^8} \times (3 + \cos^2\theta)^2, \tag{4.22}$$

where $\cos\theta$ is the scattering angle in the center-of-mass system. So far, $\gamma\gamma$ scattering for $E < m_e$ has not been observed experimentally, but there are plans to measure it using intense lasers [9].[2]

To determine the Wilson coefficients C_1, C_2, we need to perform a matching computation. It is simplest to consider the $\gamma\gamma$ scattering amplitude directly. Since we only need to extract two numbers, it suffices to evaluate the forward amplitude $\gamma(p_1) + \gamma(p_2) \to \gamma(p_1) + \gamma(p_2)$ and to consider two different helicity configurations. In QED, the amplitude is

$\mathcal{A}^{\mu_1\mu_2\nu_1\nu_2} = 2 \times$ 

$+\, 2 \times$ 

$+\, 2 \times$ 

(4.23)

The factors of two arise because an identical contribution is obtained from the diagram with reversed fermion flow ↻ vs. ↺. To get the scattering amplitude, one then has to contract with polarization vectors

$$\mathcal{A} = \mathcal{A}^{\mu_1\mu_2\nu_1\nu_2}\varepsilon_{\mu_1}\varepsilon_{\mu_2}\varepsilon^*_{\nu_1}\varepsilon^*_{\nu_2}. \tag{4.24}$$

For matching purposes, we can consider for instance

$$\mathcal{A}_1 = g^{\mu_1\mu_2}g^{\nu_1\nu_2}\mathcal{A}_{\mu_1\mu_2\nu_1\nu_2}, \qquad \mathcal{A}_2 = g^{\mu_1\nu_1}g^{\mu_2\nu_2}\mathcal{A}_{\mu_1\mu_2\nu_1\nu_2}, \tag{4.25}$$

[2] At higher energies, $\gamma\gamma$ scattering has been seen in heavy-ion collisions [10,11].

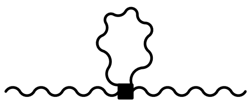
in both the full and the effective theory and then solve for C_1, C_2. The computation can be further simplified by expanding the QED diagrams in the small external momenta, which can be done on the integrand level in this case. After the expansion, the necessary integrals all have the form

$$\int d^dk \frac{(k^2)^\alpha}{(k^2 - m_e^2)^\beta}$$

and are obtained directly from Appendix A, leading to

$$C_1 = -\frac{1}{36}\alpha^2, \qquad C_2 = \frac{7}{90}\alpha^2. \tag{4.26}$$

In particular, the result for the one-loop amplitude is finite and does not contain any logarithmic corrections proportional to $\log \frac{\mu^2}{m_e^2}$. In the EFT we can directly understand the reason for this cancellation: the only loop in the EFT from $\mathcal{L}^{(8)}$ is



$\mathcal{L}^{(8)}$

$$= 0, \tag{4.27}$$

since the photon loop only contains scaleless integrals. Other loop diagrams are not possible because a second vertex from $\mathcal{L}^{(8)}$ would give an additional m_e^{-4} suppression and $\mathcal{L}^{(4)}$ and $\mathcal{L}^{(6)}$ do not contain interactions. The operators in $\mathcal{L}^{(8)}$ are thus not renormalized. Plugging in C_1 and C_2 into our earlier result for the cross section, one has

$$\frac{d\sigma_{\gamma\gamma}}{d\Omega} = 139\left(\frac{\alpha^2}{180\pi}\right)^2\left(3 + \cos^2\theta\right)^2 \frac{E^6}{m_e^8}. \tag{4.28}$$

4.2. Decoupling of heavy flavors

The quarks and leptons in the SM appear in three generations. For reasons we do not understand, the masses of the fermions are quite hierarchical $m_1 \ll m_2 \ll m_3$, e.g., $m_e = 0.5\,\mathrm{MeV}$, $m_\mu = 106\,\mathrm{MeV}$, $m_\tau = 1777\,\mathrm{MeV}$. An important generalization of the Euler–Heisenberg EFT is the EFT obtained from integrating out heavy flavors. For QED at different energies, one uses

$$\begin{array}{cc}
E \gtrsim m_\tau & \mathcal{L}_{\mathrm{QED}}[\tau, \mu, e, A] \\
 & \downarrow \\
m_\tau \gtrsim E \gtrsim m_\mu & \mathcal{L}^{\mathrm{eff}}_{\mathrm{QED}}[\mu, e, A] \\
 & \downarrow \\
m_\mu \gtrsim E \gtrsim m_e & \mathcal{L}^{\mathrm{eff}}_{\mathrm{QED}}[e, A] \\
 & \downarrow \\
m_e \gtrsim E & \mathcal{L}_{\text{Euler–Heisenberg}}[A]
\end{array}$$

Since $M_\pi \sim m_\mu$, one needs to also consider strong interaction effects once one includes the muon in the Lagrangian, but we will ignore this complication for the moment and start with

$\mathcal{L}^{\text{eff}}_{\text{QED}}[\mu, e, A]$ and construct $\mathcal{L}^{\text{eff}}_{\text{QED}}[e, A]$. We will then discuss how this Lagrangian can be used to search for physics beyond the SM and how the analog construction works in the QCD case.

4.2.1. Heavy flavors in QED

The leading-order Lagrangian $\mathcal{L}^{\text{eff}}_{\text{QED}}[e, A]$ is just the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m_e)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.29}$$

which contains the two parameters e and m_e to be determined by matching. At higher order, we get the same photonic operators as in the Euler–Heisenberg case. In addition, there are now operators containing fermion fields. Up to operator dimension $d = 6$, we have

$$\bar{\psi}\Gamma^{\mu}D_{\mu}\psi, \quad \bar{\psi}\Gamma^{\mu\nu}D_{\mu}D_{\nu}\psi, \quad \bar{\psi}\Gamma^{\mu\nu\rho}D_{\mu}D_{\nu}D_{\rho}\psi, \quad \bar{\psi}\Gamma_1\psi\bar{\psi}\Gamma_2\psi, \tag{4.30}$$

where the Γ are arbitrary Dirac matrices. At $d = 4$, the only possibility is $\bar{\psi}i\not{D}\psi$. At $d = 5$ one has

$$O_1 = \bar{\psi}\frac{1}{2}[\gamma^{\mu}, \gamma^{\nu}]D_{\mu}D_{\nu}\psi = \bar{\psi}(-i\sigma^{\mu\nu})\frac{1}{2}[D_{\mu}, D_{\nu}]\psi \tag{4.31}$$

$$= \frac{e}{2}\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi, \tag{4.32}$$

$$O_2 = \bar{\psi}\underbrace{\frac{1}{2}\{\gamma^{\mu}, \gamma^{\nu}\}}_{g_{\mu\nu}}D_{\mu}D_{\nu}\psi, \tag{4.33}$$

where we used Eq. (4.3) for the covariant derivatives. $O_1 + O_2 = \bar{\psi}\not{D}\not{D}\psi$ can be eliminated using the EOM $i\not{D}\psi = m\psi$, so we only need to consider one operator, e.g., O_1. It turns out that the Wilson coefficients of O_2 vanishes for $m_e = 0$. The reason is that $\mathcal{L}_{\text{QED}}$ has a symmetry $\psi \to e^{i\alpha\gamma_5}\psi$, $\bar{\psi} \to \bar{\psi}e^{+i\alpha\gamma_5}$ for $m_e = 0$ and O_1 violates this symmetry. So we only need to consider the $d = 6$ operator $O_{\text{mag}} = m_e\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi$. As a side remark, we note that the axial symmetry $\psi \to e^{i\alpha\gamma_5}\psi$ is *not* a symmetry of the theory, but only of the Lagrangian because the path-integral measure is not invariant. However, the measure for ψ is the same in the full and effective theory, so that this axial anomaly does not affect our argument. In addition, we have operators

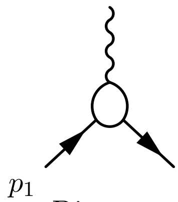
$$O_{(n)} = \bar{\psi}\Gamma_{(n)}\psi\bar{\psi}\Gamma_{(n)}\psi \tag{4.34}$$

with $\Gamma_{(n)} = \gamma^{[\alpha_1}\gamma^{\alpha_2}\dots\gamma^{\alpha_n]}$ (totally antisymmetrized) at $d = 6$. Because of axial symmetry, the terms with even n will have coefficients $\propto m_e$. The operators with three covariant derivatives all reduce to O_2', $O_{(1)}$, and $O_{(3)}$. In particular

$$\bar{\psi}\partial_{\sigma}F_{\mu\nu}\gamma^{[\mu}\gamma^{\nu}\gamma^{\sigma]}\psi = 0 \tag{4.35}$$

from the Bianchi identity. Accordingly, we conclude that, up to terms suppressed by at least m_{μ}^{-3}, all effects of physics at scales $E \ll m_{\mu}$ can be absorbed into the electron mass and the electromagnetic coupling, as well as the Wilson coefficients of O_{mag}, $O_{(1)}$, and $O_{(3)}$.

Let us now discuss the physics associated with O_{mag} and how the effects of physics beyond the Standard Model manifests themselves in C_{mag}. For this purpose, we consider the interaction of an electron with a background electromagnetic field



$$= \bar{u}(p_2)\Gamma^\mu(p_1,p_2)u(p_1) \times (-ieA_\mu), \tag{4.36}$$

where the independent Dirac structures are

$$\Gamma^\mu = A\gamma^\mu + B(p_1+p_2)^\mu + C(p_1-p_2)^\mu + A'\gamma^\mu\gamma^5 + B'(p_1+p_2)^\mu\gamma^5 + C'(p_1-p_2)^\mu\gamma^5. \tag{4.37}$$

The coefficients A', B', C' are zero because of parity invariance of QED. Additional structures involving $\not{p}_1$ or $\not{p}_2$ can be eliminated using the EOM $\not{p}u(p) = m_e u(p)$. Furthermore, current conservation implies

$$q^\mu \bar{u}(p_2)\Gamma_\mu u(p_1) = 0 = Cq^2 \Rightarrow C = 0. \tag{4.38}$$

We are thus left with two functions $A(q^2)$ and $B(q^2)$. It is customary to write Γ^μ in the form

$$\Gamma^\mu = F_1(q^2)\gamma^\mu + \frac{i}{2m_e}\sigma^{\mu\nu}q_\nu F_2(q^2), \qquad q = p_2 - p_1. \tag{4.39}$$

At tree level in the EFT, we have $F_1(q^2) = 1$ and $F_2(q^2) = C_{\text{mag}} m_e^2$, where we wrote

$$\mathcal{L}_{\text{mag}} = \frac{C_{\text{mag}}}{4} O_{\text{mag}} = \frac{C_{\text{mag}}}{4} m_e \bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi, \tag{4.40}$$

while in QED $F_1 = 1$ already at tree level as well, while F_2 is only generated by loop contributions. To understand the meaning of F_1 and F_2, let us consider the non-relativistic limit $\mathbf{p}_1$, $\mathbf{p}_2 \to 0$. In the basis $\gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}$, $\gamma^i = \begin{pmatrix} & \sigma^i \\ \sigma^i & \end{pmatrix}$, one has

$$u(p) = \sqrt{p^0 + m_e}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m_e}\chi_s \end{pmatrix} = \sqrt{2m_e}\begin{pmatrix} \chi_s \\ 0 \end{pmatrix} + \mathcal{O}(\mathbf{p}). \tag{4.41}$$

Moreover, using the Gordon identity

$$\bar{u}(p_2)\gamma^\mu u(p_1) = \bar{u}(p_2)\left[\frac{(p_1+p_2)^\mu}{2m_e} + \frac{i}{2m_e}\sigma^{\mu\nu}q_\nu\right]u(p_1) \tag{4.42}$$

we can replace the γ^μ term in favor of $(p_1+p_2)^\mu$, which is easier to handle in the non-relativistic expansion

$$A_\mu \frac{(p_1+p_2)^\mu}{2m_e}\bar{u}(p_2)u(p_1) = 2m_e A^0 \chi_s^\dagger \chi_s + \mathcal{O}(\mathbf{p}^2), \tag{4.43}$$

and

$$A_\mu \bar{u}(p_2)\frac{i}{2m_e}\sigma^{\mu\nu}q_\nu u(p_1) \cong A_i \bar{u}(p_2)\frac{-1}{4m_e}[\gamma^i,\gamma^j]q_j u(p_1) \tag{4.44}$$

$$\cong -\frac{A_i}{4m_e}2m_e\chi_s^\dagger\sigma^k\chi_s 2i\epsilon^{ijk}q_j \tag{4.45}$$

$$= -iA_i q_j \chi_s^\dagger \sigma_k \chi_s \epsilon^{ijk} \tag{4.46}$$

$$= -\chi_s^\dagger \boldsymbol{\sigma}\cdot\mathbf{B}(q)\chi_s, \tag{4.47}$$

where $B_k(q) = -i\epsilon^{ijk} q_i A_j(q) \hat{=} (\boldsymbol{\nabla} \times \mathbf{A}(q))_k$ and we used $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$. The QM Hamiltonian describing the interaction of an electron with an electromagnetic field contains a term

$$\mathbb{H} = -g_e \frac{e}{2m_e} \mathbf{S} \cdot \mathbf{B} = -\boldsymbol{\mu} \cdot \mathbf{B}. \tag{4.48}$$

For an electron $\mathbf{S} = \frac{\boldsymbol{\sigma}}{2}$, and comparing with our expression for $\bar{u}\Gamma^\mu u(-ieA_\mu)$ (while accounting for the spinor normalization $2m_e$), we find

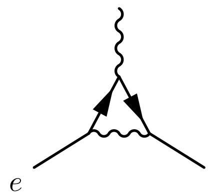
$$g_e = 2[F_1(0) + F_2(0)] = 2 + 2F_2(0). \tag{4.49}$$

The deviation of the gyromagnetic ratio g_e from 2 is called the anomalous magnetic moment

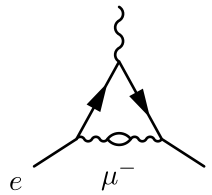
$$a_e = \frac{g_e - 2}{2}. \tag{4.50}$$

It receives contributions from quantum corrections and from the operator O_{mag}, whose Wilson coefficient encapsulates the contribution from heavier states. Because it is sensitive to such corrections from heavier states, precision measurements of anomalous magnetic moments are used to search for physics beyond the SM.

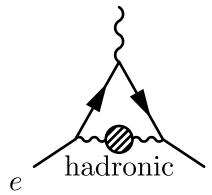
Let us take a look at the different contributions to a_e. First, there are QED and hadronic contributions:



$\rightarrow \quad a_e = \frac{\alpha}{2\pi} = 10^{-3}$ [12] (Even the $\mathcal{O}(\alpha^5)$ corrections are known [13]!)

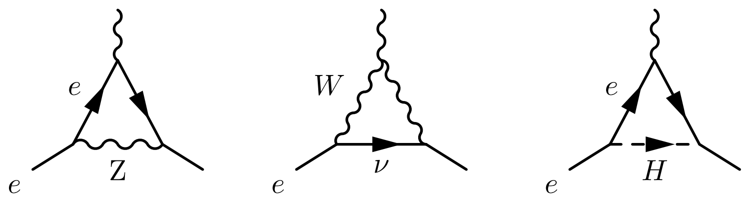


$\rightarrow \quad \frac{1}{45}\frac{m_e^2}{m_\mu^2}\left(\frac{\alpha}{\pi}\right)^2$



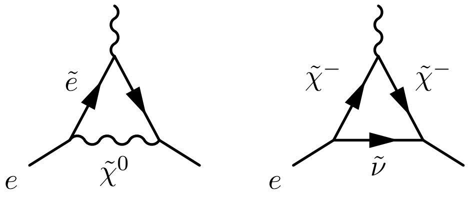
$\rightarrow \quad m_e^2 \triangle C_{\text{mag}}^{\text{hadronic}}$ (obtained from experimental data)

Next, there are electroweak corrections:



$\propto \frac{\alpha}{\pi}\frac{m_e^2}{M_W^2}$

In addition to SM contributions, there could also be physics beyond the SM, e.g., supersymmetry (SUSY):



$$\propto \frac{\alpha}{\pi}\frac{m_e^2}{\tilde{m}^2}$$

These contributions decouple with the heavy mass scale $\tilde{m}$, but could lead to deviations from the SM prediction that can be detected in experiment.

Since $(m_e/M_W)^2 \sim 4 \times 10^{-11}$, the weak effects are very small for the electron $g-2$. The effect of higher-mass particles are often enhanced by $(\frac{m_\mu}{m_e})^2 \approx 4 \times 10^4$ in the *muon* $g-2$. The current world average after the release of Run 1 data from the Fermilab E989 experiment [14] differs from the SM prediction [15] by 4.2σ, providing hints for contributions beyond the SM.

4.2.2. Heavy flavors in QCD

The quark masses are very hierarchical and for many application one will need to integrate out the heavy flavors $m_t \simeq 173\,\text{GeV}$ and $m_b \simeq 5\,\text{GeV}$. The masses of these quarks are large enough that the matching can be performed perturbatively. For the charm quark with $m_c \simeq 1.3\,\text{GeV}$, $\alpha_s(m_c) \simeq 0.32$, this is also true, but the corrections will be significant. Let us first discuss $\mathcal{L}_{\text{eff}}$ up to dimension 6 and then the matching for the $d=4$ Lagrangian. All the operators found in QED are also allowed in QCD. The effective QCD Lagrangian after integrating out the top quark has the form

$$\mathcal{L}_{d=4} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} + \sum_{f=1}^{5}\bar{\psi}_f(i\not{D} - m_f)\psi_f. \tag{4.51}$$

This is simply QED with 5 flavors.

For $d=6$, we have the operators

$$O^{ff'}_{(5i)} = \bar{\psi}_f\gamma^{[\mu_1}\dots\gamma^{\mu_i]}\psi_f\bar{\psi}_{f'}\gamma_{[\mu_1}\dots\gamma_{\mu_i]}\psi_{f'}, \tag{4.52}$$

$$O^{ff'}_{(0i)} = \bar{\psi}_f t^a\gamma^{[\mu_1}\dots\gamma^{\mu_i]}\psi_f\bar{\psi}_{f'}t^a\gamma_{[\mu_1}\dots\gamma_{\mu_i]}\psi_{f'}. \tag{4.53}$$

Since QCD does not distinguish the flavors,[3] the Wilson coefficients of $O^{ff'}_{(5i)}$, $O^{ff'}_{(0i)}$ are independent of the flavor indices, i.e., we only need the two operators

$$O_{(5i)} = \sum_{ff'}O^{ff'}_{(5i)}, \qquad O_{(0i)} = \sum_{ff'}O^{ff'}_{(0i)}. \tag{4.54}$$

As in QED, we get

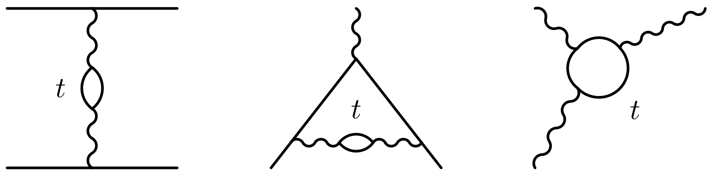
$$O_{\text{mag}} = \sum_f m_f\bar{\psi}_f\sigma^{\mu\nu}G^a_{\mu\nu}t^a\psi_f, \tag{4.55}$$

but there is one additional operator, namely

$$O_3 = f^{abc}G^a_{\mu\nu}G^{\nu\sigma}_bG^{c,\mu}_\sigma, \tag{4.56}$$

[3] Except for the quark masses, which can be set to zero for the matching computation

which arises from the non-Abelian nature of QCD. The leading contributions to the Wilson coefficients of $O_{(01)}$ and O_3 originate from



Let us now discuss the matching for $\mathcal{L}_{d=4}$, which contains as Wilson coefficient the coupling constant g_s and m_f, and concentrate on $\alpha_s(\mu) = \frac{g_s^2(\mu)}{4\pi}$. We denote the coupling by $\alpha_s^{(n_f)}(\mu)$ to distinguish α_s in the theory with $n_f = 6$ from $\alpha_s^{(5)}(\mu)$ in the theory where the top quark is integrated out. Then we proceed as in our scalar toy model. At a scale $\mu_m \approx m_t$ one derives a relation

$$\alpha_s^{(5)}(\mu_m) = \alpha_s^{(6)}(\mu_m)\xi_A^{-1}\big[\alpha_s^{(6)}(\mu_m)\big]. \tag{4.57}$$

The simplest way to obtain ξ_A is to compute the gluon propagator in both theories ("os" refers to "on-shell" scheme):

$$G_{\mu\nu} = \frac{iZ_A^{\text{os}}}{p^2}(-g_{\mu\nu} + \dots). \tag{4.58}$$

Rescaling the coupling by ξ_A is the same as rescaling the gluon field. One can then show that

$$\xi_A^{(0)} = \frac{Z_A^{(6)}}{Z_A^{(5)}} \quad \text{with} \quad Z_A = \frac{1}{1 - \Pi(0)}, \tag{4.59}$$

where $\Pi(0)$ is the polarization function evaluated at virtuality $q^2 = 0$. If one chooses $\mu_m = m_t(\mu_m)$ the expression for ξ_A is especially simple

$$\xi_A(m_t) = 1 + \left(\frac{13}{3}C_F - \frac{32}{9}C_A\right)T_F\left(\frac{\alpha(m_t)}{4\pi}\right)^2, \tag{4.60}$$

where $C_F = (N_c^2 - 1)/(2N_c) = 4/3$, $C_A = N_c = 3$ are the Casimir invariants of the fundamental and adjoint representations, respectively, and $T_F = 1/2$ is the normalization of the generators.

Note that the QCD coupling runs differently for $n_f = 5$ and $n_f = 6$:

$$\mu\frac{d\alpha_s(\mu)}{d\mu} = \beta(\alpha_s(\mu)), \tag{4.61}$$

$$\beta(\alpha_s) = -2\alpha_s\left[\beta_0\left(\frac{\alpha_s}{4\pi}\right) + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \dots\right], \tag{4.62}$$

with $\beta_0 = (11C_A - 4n_fT_F)/3$ and $\beta_1 = 2\big[17C_A^2 - n_fT_F(10C_A + 6C_F)\big]/3$.

4.3. Effective weak Hamiltonian

Let us now discuss the weak interactions at low energies. In the SM, the weak and electromagnetic interactions are described by a $SU(2)_L \times U(1)_Y$ gauge theory, which is broken down to $U(1)_{\text{em}}$ by the Higgs mechanism, which gives masses to the $W^\pm$ and Z^0 bosons. A detailed introduction to the SM is beyond the scope of this lecture. The only information needed for our discussion is the charged-current coupling of $W^\pm$ bosons to fermions. It has the form

$$\mathcal{L}_{\text{cc}} = -\frac{g_2}{2\sqrt{2}}(J_\mu^+ W^{+\mu} + J_\mu^- W^{-\mu}), \tag{4.63}$$

where

$$J_\mu^+ = (J_\mu^-)^\dagger = (\bar{u}d')_{V-A} + (\bar{c}s')_{V-A} + (\bar{t}b')_{V-A} + (\bar{\nu}_e e)_{V-A} + (\bar{\nu}_\mu \mu)_{V-A} + (\bar{\nu}_\tau \tau)_{V-A} \tag{4.64}$$

and $(\bar{u}d')_{V-A} = \bar{u}\gamma_\mu(1-\gamma_5)d'$, etc. The states d', s', b' are not mass eigenstates, i.e., the quadratic part of $\mathcal{L}_{\text{SM}}$ is not diagonal. The CKM[4] matrix connects (d', s', b') to the mass eigenstates (d, s, b)

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = V_{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{4.65}$$

The matrix is unitary, so $V_{\text{CKM}} V_{\text{CKM}}^\dagger = \mathbb{1}$. It can be further simplified by phase redefinitions of the fermion fields in $\mathcal{L}_{\text{cc}}$ and has $3 \times 3 - (2 \times 3 - 1) = 4$ physical parameters. Its structure is reflected by the Wolfenstein parameterization, which was designed to show the hierarchy of the different matrix elements:

$$\hat{V}_{\text{CKM}} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \tag{4.66}$$

with

$$\lambda \simeq 0.225, \quad \rho \simeq 0.139, \quad A \simeq 0.81, \quad \eta \simeq 0.342.$$

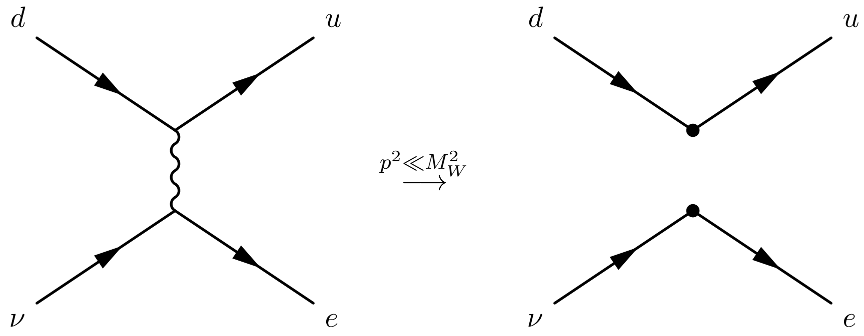
This form is only approximately unitary, up to higher orders in λ. The parameters of the matrix correspond to three rotations and one complex phase, which leads to CP violation. Similarly, the neutrinos ν_e, ν_μ, and ν_τ are not mass eigenstates. The corresponding mixing matrix is called PMNS matrix (Pontecorvo–Maki–Nakagawa–Sakala), but will not play a role in what follows.

Let us first work at tree level and neglect QCD effects. Then the effective weak Lagrangian can be obtained by integrating out the $W^\pm$ and Z fields. The resulting effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = -\frac{g_2^2}{8M_W^2}\left[J_\mu^- J^{+\mu} + \frac{1}{M_W^2} J_\mu^- (\partial^\mu\partial^\nu - g^{\mu\nu}\Box) J_\nu^+ + \dots\right] \tag{4.67}$$

where $\frac{g_2^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}$ defines the Fermi constant $G_F = 1.166 \times 10^{-5}\,\text{GeV}^{-2}$. Diagrammatically, this arises from

[4]Cabibbo–Kobayashi–Maskawa matrix, Nobel Prize in 2008.



where the W-propagator is expanded as

$$\frac{-i}{p^2 - M_W^2}\left[g^{\mu\nu} - \frac{p^\mu p^\nu}{M_W^2}\right] = \frac{i}{M_W^2}\left[g^{\mu\nu} - \frac{1}{M_W^2}(p^\mu p^\nu - p^2 g^{\mu\nu}) + \mathcal{O}\left(\frac{1}{M_W^4}\right)\right], \tag{4.68}$$

so that already the leading terms in $\mathcal{L}_{\text{eff}}$ are irrelevant operators of $d = 6$ (recall that the fermion field has $d = 3/2$). Indeed the coefficient of the four-fermion operators $G_F \sim \frac{1}{M_W^2}$ shows the expected behavior. The fact that these are not marginal or relevant operators explains the apparent weakness of the interaction at low energies. At high energies, on the other hand, the weak-interaction effects are as strong as electromagnetic interactions. Because of the $\frac{1}{M_W^2}$ suppression the leading-order $d = 6$ terms are good enough for most applications. Since it changes lepton and quark flavors, $\mathcal{L}_{\text{eff}}$ governs all decays of heavy leptons and hadrons, such as

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu, \tag{4.69}$$
$$\pi^- \to \mu^- + \bar{\nu}_\mu, \tag{4.70}$$
$$n \to p + e^- + \bar{\nu}_e. \tag{4.71}$$

The PDG lists hundreds of pages of various hadron decays. In the SM all such decays that proceed via the weak interactions are governed by G_F and the four parameters in the CKM matrix. If one manages to evaluate the strong-interaction effects in such decays, they offer many opportunities to search for physics beyond the SM. An important step is to include QCD corrections to the effective Lagrangian. To do so, one has to

1. include a complete set of operators, not only those present at tree level,
2. perform a matching computation to obtain the Wilson coefficients,
3. solve the RG equation for the coefficients to resum large logs.

We will now discuss two examples that illustrate how the construction works in the general case.

4.3.1. Leptonic decays

Let us consider the operator relevant for $\pi^- \to \mu^- + \bar{\nu}_\mu$, which is based on the quark-level transition $\bar{u}d \to \bar{\nu}_\mu \mu^-$. The tree-level $\mathcal{L}_{\text{eff}}$ is

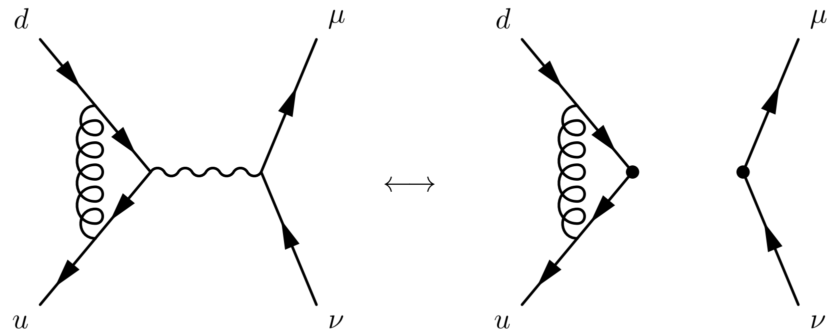
$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{ud} (\bar{u}d)_{V-A} (\bar{\mu}\nu_\mu)_{V-A}. \tag{4.72}$$

It turns out that this is the only operator with this flavor structure: the weak interactions only couple to left-handed fields $\psi_L = \frac{1}{2}(1-\gamma_5)\psi$ and the QCD interactions conserve helicity for vanishing quark masses $\mathcal{L}_{\rm QCD} = \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{D}\psi_R$. Chirality violating operators are suppressed by powers of the quark masses. The only possible Dirac bilinears are $\bar{\psi}_L\gamma^\mu\psi_L = -\bar{\psi}_L\gamma^\mu\gamma^5\psi_L$ and $\bar{\psi}_L\psi_L = \bar{\psi}_L\sigma^{\mu\nu}\psi_L = 0$. This leaves as operators only

$$\bar{u}_L\gamma^\mu d_L\,\bar{\mu}_L\gamma_\mu\nu_L = \bar{u}_L\gamma^\mu\nu_L\,\bar{\mu}_L\gamma_\mu d_L. \tag{4.73}$$

The fact that the two operators are equal is an example of Fierz identities, which follow from a rearrangement of the Dirac structures.

Not only is there just a single operator, but also the matching is trivial since all QCD corrections are the same in the full and the effective theory:



Accordingly, there are no QCD corrections to $\mathcal{L}_{\rm eff}$. QCD effects only arise in the matrix element

$$\langle 0|\,\bar{u}\gamma^\mu(1-\gamma_5)d\,\big|\pi^-(p)\big\rangle = -\,\langle 0|\,\bar{u}\gamma^\mu\gamma_5 d\,\big|\pi^-(p)\big\rangle \tag{4.74}$$

$$= -if_\pi p^\mu = -i\sqrt{2}F_\pi p^\mu. \tag{4.75}$$

The matrix element F_π is called the pion decay constant. By measuring the pion decay rate

$$\Gamma(\pi\to\mu\nu) = \frac{G_F^2}{4\pi}F_\pi^2 m_\mu^2 M_\pi\left(1-\frac{m_\mu^2}{M_\pi^2}\right)|V_{ud}|^2 \tag{4.76}$$

one can then determine the combination $|V_{ud}|F_\pi$. Similarly, one obtains $|V_{cd}|$ from $D^-\to\mu^-\bar{\nu}$, $|V_{ub}|$ from $B^-\to\tau^-\bar{\nu}$, and $|V_{us}|$ from $K^-\to\mu^-\bar{\nu}$, in combination with the respective decay constants.

4.3.2. Hadronic decays

Let us consider next the decay $\bar{B}^{(0)}\to D_s^-\pi^+$, based on the $b\to u\bar{c}s$ quark-level transition. In this case there are two operators that differ by their color structure

$$O_1 = \bar{s}_L^i\gamma_\mu c_L^i\bar{u}_L^j\gamma^\mu b_L^j, \tag{4.77}$$

$$O_2 = \bar{s}_L^i\gamma_\mu c_L^j\bar{u}_L^j\gamma^\mu b_L^i, \tag{4.78}$$

where the color indices i and j are summed over. Note that

$$\bar{s}_L\gamma_\mu t^a c_L\bar{u}_L\gamma^\mu t^a b_L = \frac{1}{2}O_2 - \frac{1}{2N_c}O_1, \tag{4.79}$$

which follows from the identity

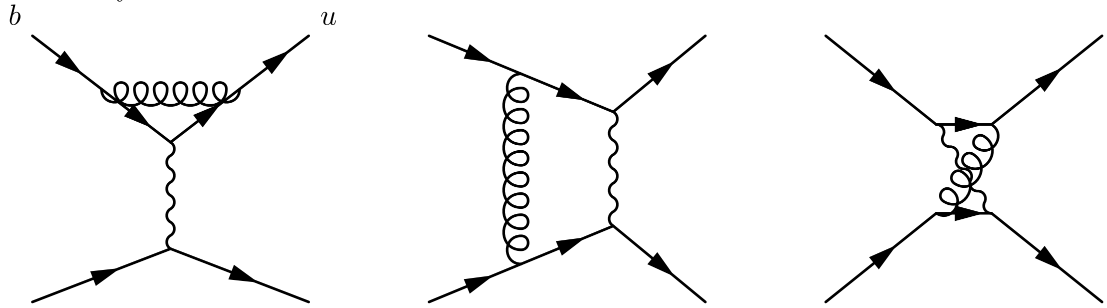
$$t^a_{ij} t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) \tag{4.80}$$

for the $SU(3)_c$ generators. As we will see, the matching in this case is now non-trivial and we write

$$\mathcal{L}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V^*_{cs} V_{ub} \Big[C_1(\mu) O_1 + C_2(\mu) O_2\Big]. \tag{4.81}$$

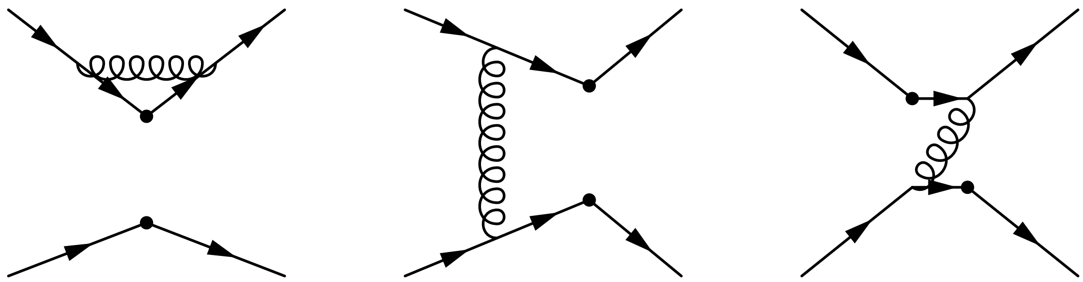
At tree level one has $C_1 = 1$, $C_2 = 0$, which follows by the same argument as for the leptonic example discussed before. To obtain the one-loop coefficients, however, one has to perform a matching computation, i.e., one has to compute the quark currents in both the full and the effective theory.

Full theory:



(a) (b) (c)

EFT:



(a) (b) (c)

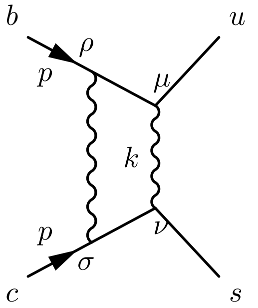
+ "mirrored" diagrams

The difference between the full and the effective-theory results is absorbed into C_1 and C_2. Since C_1, C_2 are independent of m_q, we can set all quark masses to zero. Furthermore, also any values for the external momenta will work, with the simplest choice $p_i = 0$. In this case diagram (a) in the full theory and all diagrams in the effective theory vanish, because they are scaleless

$$\int d^d k \frac{1}{k^2}\frac{1}{\not{k}}\Gamma\frac{1}{\not{k}}\Gamma = 0 = \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}}, \tag{4.82}$$

which amounts to a cancellation of IR and UV divergences in dimensional regularization. Since the IR divergences are a low-energy property of the theory, they are present in both the full and the effective theory, and thus cancel in the matching. For pedagogical reasons we will keep the p_i non-zero, but use the same value for all legs. Let us start with diagram (b) in the full theory

$(b) =$



(4.83)

$$
\begin{aligned}
&= i^2 \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ub} \int \frac{d^d k}{(2\pi)^d} \frac{-i}{k^2} \frac{-i}{k^2 - M_W^2} \frac{i}{(p+k)^2} \frac{i}{(p-k)^2} \\
&\times \bar{u}\gamma^\mu (1-\gamma_5)(\not{p} + \not{k}) i g_s t^a \gamma^\rho b \times \bar{s}\gamma^\nu (1-\gamma_5)(\not{p} - \not{k}) i g_s t^a \gamma^\sigma c \\
&\times g_{\rho\sigma} \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2} \right].
\end{aligned}
\tag{4.84}
$$

Some remarks are in order:

1. This is an amputated Green's function, the spinors $\bar{u}\Gamma_1 b\, \bar{s}\Gamma_2 c$ are only there to remind ourselves which color and spin index goes where.

2. We are using Feynman gauge for QCD and unitary gauge for the W-propagator. The $k^\mu k^\nu / M_W^2$ term does not contribute to the sum of the diagrams, so we will omit it.

3. The color structure is $t^a \otimes t^a$. To rewrite this in the form of O_1 and O_2, we use the identity (4.80).

4. To simplify the Dirac structure one needs identities such as $[\Gamma = \gamma^\alpha (1-\gamma^5)]$

$$
\Gamma \gamma_\beta \gamma_\mu \otimes \Gamma \gamma^\beta \gamma^\mu = 16 \Gamma \otimes \Gamma. \tag{4.85}
$$

   The coefficient can be derived by taking traces $\mathrm{Tr}[A\Gamma\gamma_\beta\gamma_\mu B\Gamma\gamma^\beta\gamma^\mu]$, $\mathrm{Tr}[A\Gamma B\Gamma]$ for some Dirac matrices A and B, e.g., $A = B = \gamma^5$.

5. Without the $k^\mu k^\nu / M_W$ term, the diagram is finite, so we only need its value for $d = 4$.

The diagram (b) in the effective theory, on the other hand, is divergent, as it behaves like

$$
\int d^d k \frac{1}{k^2} \frac{1}{\not{p} - k} \frac{1}{\not{p} + k} \sim \int \frac{d^d k}{k^4} \sim \frac{1}{\epsilon} + \log \frac{\mu^2}{p^2}. \tag{4.86}
$$

This leads to some technical issues:

1. The Dirac basis can be thought of as all totally antisymmetric products of γ matrices. In d dimensions, one can also write down antisymmetric products of more than four γ matrices, which are not necessary in four dimensions. Operators with such Dirac structures are called *evanescent*. One can use a renormalization scheme in which their physical matrix elements vanish, but they need to be included in $\mathcal{L}_{\text{eff}}$ for consistency.

2. γ^5 is special to $d = 4$. The rule $\{\gamma^\mu, \gamma^5\} = 0$ leads to inconsistencies in d dimensions. A prescription to eliminate γ^5 is the 't Hooft/Veltman scheme [16], which can be thought of as replacing $\gamma_5 = -\frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, contracting two epsilon tensors in terms of metric tensors, and only dealing with a potentially remaining epsilon tensor *after* renormalization. Alternatively, one can use the so-called NDR scheme (naive dimensional regularization), i.e., use $0 = \{\gamma^\mu, \gamma^5\}$ and hope that no inconsistency arise, or the DRED scheme (dimensional reduction), i.e., treat γ^μ and the gauge fields as four dimensional.

Let us now discuss the results. The bare Green's function in the SM and in the EFT are ($d = 4 - 2\epsilon$):

$$\begin{aligned}\Gamma_{\text{full}} = \frac{G_F V_{cs}^* V_{ud}}{\sqrt{2}} \Bigg\{ &\left(1 + 2C_F \frac{\alpha_s}{4\pi}\Big(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\Big)\right) \langle O_1\rangle_{\text{tree}} + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{-p^2}\langle O_1\rangle_{\text{tree}} \\ &- 3\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{-p^2}\langle O_2\rangle_{\text{tree}}\Bigg\} + \mathcal{O}\Big(\frac{p^2}{M_W^2}\Big),\end{aligned} \tag{4.87}$$

$$\begin{aligned}\Gamma_{\text{eff}} = \frac{G_F V_{cs}^* V_{ud}}{\sqrt{2}} \Bigg\{ &C_1^{\text{bare}}\Bigg[\left(1 + 2C_F \frac{\alpha_s}{4\pi}\Big(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\Big)\right) \langle O_1\rangle_{\text{tree}} \\ &+ \frac{3}{N_c}\frac{\alpha_s}{4\pi}\Big(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\Big)\langle O_1\rangle_{\text{tree}} - 3\frac{\alpha_s}{4\pi}\Big(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\Big)\langle O_2\rangle_{\text{tree}}\Bigg] \\ + C_2^{\text{bare}}\Bigg[&\left(1 + 2C_F \frac{\alpha_s}{4\pi}\Big(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\Big)\right) \langle O_2\rangle_{\text{tree}} + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\Big(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\Big)\langle O_2\rangle_{\text{tree}} \\ &- 3\frac{\alpha_s}{4\pi}\Big(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\Big)\langle O_1\rangle_{\text{tree}}\Bigg]\Bigg\} + \mathcal{O}\Big(\frac{p^2}{M_W^2}\Big).\end{aligned} \tag{4.88}$$

In each case, the first term corresponds to diagram (a), the remainder to (b)+(c) (and their mirrored counterparts). The color factors are $N_c = 3$ and $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$, $\alpha_s = \frac{g_s^2}{4\pi}$. The Wilson coefficients have an expansion $C_i = C_i^{(0)} + \frac{\alpha_s}{4\pi}C_i^{(1)} + \ldots$, as reflected by the α_s corrections in Eq. (4.87). The bare amputated Green's functions have divergences, corresponding to the $1/\epsilon$ poles. In the full theory, these are removed by the wave-function renormalization $\psi^{(0)} = Z_q^{1/2}\psi$, with the one-loop wave function renormalization $Z_q = 1 - \frac{\alpha_s}{4\pi\epsilon}C_F$ in $\overline{\text{MS}}$ and for Feynman gauge. In the effective theory, there are additional divergences from (b) and (c), which are not removed by wave-function renormalization of the Wilson coefficients in $\mathcal{L}_{\text{eff}}$. Omitting the overall factor $-\frac{G_F}{\sqrt{2}}V_{cs}^* V_{ud}$, one has

$$\mathcal{L}_{\text{eff}} = C_i^{\text{bare}} O_i(q^{(0)}) = Z_q^2 C_i Z_{ij} O_j(q). \tag{4.89}$$

The additional renormalization constants form a matrix Z_{ij}, and expanding $Z_{ij} = \delta_{ij} + \frac{\alpha_s}{4\pi\epsilon}Z_{ij}^{(1)}$, one finds that

$$Z = \mathbb{1} + \frac{\alpha_s}{4\pi\epsilon}\begin{pmatrix} -3/N_c & 3 \\ 3 & -3/N_c \end{pmatrix} \tag{4.90}$$

removes the remaining divergence in Γ_{eff}.

From the condition $\Gamma_{\text{full}} - \Gamma_{\text{eff}} = 0$, one can then finally read off C_1 and C_2:

$$C_1(\mu) = 1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{\mu^2}, \tag{4.91}$$

$$C_2(\mu) = 0 - \frac{3}{N_c}\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{\mu^2}. \tag{4.92}$$

The crucial point here is that C_1 and C_2 only depend on M_W and μ, but not on the low-energy scale p^2. This has to be the case, as the low-energy physics must drop out in the matching. The last step in the construction of $\mathcal{L}_{\text{eff}}$ is to solve the RG equations for C_1 and C_2 to avoid having large logarithms when evaluating $C_1(\mu)$ at low values of μ. The RG equations follow from the fact that physical quantities are μ independent. Equivalently, we can use the fact that bare quantities are μ independent:

$$\mu\frac{d}{d\mu}C_j^{\text{bare}}(\epsilon,\alpha^{\text{bare}}) = 0 = \mu\frac{d}{d\mu}C_i(\mu)Z_{ij}(\alpha_s(\mu),\epsilon) \tag{4.93}$$

$$= \left(\mu\frac{d}{d\mu}C_i(\mu)\right)Z_{ij} + C_i(\mu)\left(\mu\frac{d}{d\mu}Z_{ij}\right), \tag{4.94}$$

so that

$$\mu\frac{d}{d\mu}C_j(\mu) - C_i(\mu)\gamma_{ij}(\alpha) = 0, \tag{4.95}$$

with $\gamma_{ij} = -(\mu\frac{d}{d\mu}Z_{ik})Z_{kj}^{-1}$, or, in vector notation,

$$\left(\mu\frac{d}{d\mu} - \hat{\gamma}^T\right)\vec{C}(\mu) = 0. \tag{4.96}$$

In the $\overline{\text{MS}}$ scheme, the Z-matrix is a sum of pole terms $\hat{Z} = \mathbb{1} + \sum_{k=1}^{\infty}\frac{1}{\epsilon^k}\hat{Z}_k(\alpha_s)$ and there is a simple relation ("magic relation")

$$\hat{\gamma} = 2\alpha_s\frac{\partial\hat{Z}_1}{\partial\alpha_s} = \frac{\alpha_s}{4\pi}\begin{pmatrix}-6/N_c & 6\\ 6 & -6/N_c\end{pmatrix}. \tag{4.97}$$

To solve the RG equation, it is simplest to use a basis in which $\hat{\gamma}$ is diagonal, in our case the corresponding combinations are $C_\pm = C_1 \pm C_2$, with

$$\mu\frac{d}{d\mu}C_\pm(\mu) = \frac{\alpha_s}{4\pi}6\Big(\pm 1 - \frac{1}{N_c}\Big)C_\pm(\mu) \tag{4.98}$$

$$\equiv -\frac{\alpha_s}{4\pi}\gamma_\pm C_\pm(\mu). \tag{4.99}$$

Converting the derivatives using the QCD β function

$$\mu\frac{d\alpha_s}{d\mu} = -2\alpha_s\left[\frac{\alpha_s}{4\pi}\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2\beta_1 + \cdots\right], \tag{4.100}$$

we can solve these RG equations by a separation of variables

$$\frac{dC_\pm}{C_\pm} = -\,d\log\mu\frac{\alpha_s}{4\pi}\gamma_\pm = -\frac{d\alpha_s}{\beta(\alpha_s)}\frac{\alpha_s}{4\pi}\gamma_\pm = \frac{d\alpha_s}{\alpha_s}\frac{\gamma_\pm}{2\beta_0} + \cdots, \tag{4.101}$$

with the result

$$\log \frac{C_\pm(\mu)}{C_\pm(M_W)} = \frac{\gamma_\pm}{2\beta_0} \log \frac{\alpha_s(\mu)}{\alpha_s(M_W)} \tag{4.102}$$

or

$$C_\pm(\mu) = C_\pm(M_W) \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{\frac{\gamma_\pm}{2\beta_0}} . \tag{4.103}$$

Using $C_\pm(M_W) = 1 + \mathcal{O}(\alpha_s)$ and rotating back to the original basis we find

$$C_1(\mu) = \frac{1}{2} \left[\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{\frac{\gamma_+}{2\beta_0}} + \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{\frac{\gamma_-}{2\beta_0}} \right], \tag{4.104}$$

$$C_2(\mu) = \frac{1}{2} \left[\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{\frac{\gamma_+}{2\beta_0}} - \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)} \right)^{\frac{\gamma_-}{2\beta_0}} \right], \tag{4.105}$$

where

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f, \qquad \gamma_\pm = -6 \Big(\pm 1 - \frac{1}{N_c} \Big) = \mp 6 + 2. \tag{4.106}$$

Numerically, one finds for $\mu = m_b$

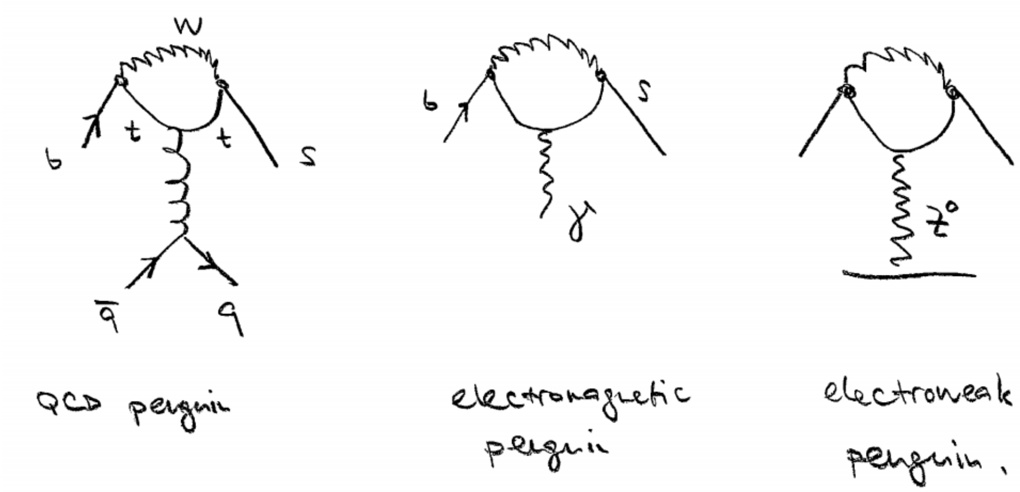
$$C_1(\mu) = 1.10, \quad C_2(\mu) = -0.24, \tag{4.107}$$

so that at the low scale indeed a non-vanishing result for C_2 has been induced, while the value of C_1 changes by a similar amount. This completes our discussion the QCD effects in $\mathcal{L}_{\rm eff}$ relevant for $b \to u\bar{c}s$, which mediates $\bar{B} \to D_s^- \pi^+$.

The structure of $\mathcal{L}_{\rm eff}$ becomes more complicated for flavor-changing-neutral current (FCNC) processes such as

$$b \to s\gamma, \qquad b \to sg, \qquad b \to s\bar{q}q, \qquad b \to s l^+ l^-. \tag{4.108}$$

For such decays so-called penguin diagrams contribute, e.g.,



Such processes are interesting because they can violate CP symmetry and are sensitive to new physics because they only arise at loop-level in the SM. The effective Lagrangian for such processes contains ~ 12 operators, which all mix under renormalization, i.e.,

$$\left[\mu\frac{d}{d\mu} - \hat{\gamma}^T\right]\vec{C} = 0. \tag{4.109}$$

Although there is a lot of interesting phenomenology associated with such decays, we will not discuss them further, but let us note that the corresponding anomalous-dimension matrix has been calculated at $\mathcal{O}(\alpha_s^3)$, which involved the computation of hundreds of thousands of three- and four-loop diagrams.

4.4. Chiral perturbation theory

Let us finally turn to the strong interaction at low energies. Instead of quarks and gluons, the observed particles are hadrons, i.e., mesons such as π, K, η, η', ρ, ... and baryons p, n, Δ, Σ, ... The effective Lagrangian is then a function of hadron fields. As in all our previous applications, one starts by writing down the most general $\mathcal{L}_{\text{eff}}$ compatible with the symmetries of the underlying theory, i.e., QCD. In contrast to previous examples, however, we will be unable to perform matching computations due to our limited ability to perform QCD computations at low energy (using lattice-QCD simulations, it is becoming possible to some extent). At first sight, it looks like an effective theory of hadrons and it will be not very predictive since the Wilson coefficients are not known. However, it turns out that chiral symmetry severely constrains the interactions of the light hadrons, and the EFT approach is very useful to derive the consequences of this approximate symmetry.

4.4.1. Chiral Symmetry

Since we will work at very low energies, we can integrate out the heavy-quark flavors and use

$$\mathcal{L}_{\text{QCD}}^{\text{eff}} = -\frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu} + \sum_{f=u,d,s}\bar{\psi}_f(i\not{D} - m_f)\psi_f + \mathcal{O}\Big(\frac{1}{m^2_{c,b,t}}\Big). \tag{4.110}$$

The theory simplifies further in the *chiral limit* $m_q \to 0$. Since only the mass term distinguishes different flavors, a new flavor symmetry arises. In fact, the symmetry group is even larger: splitting $\psi_{L,R} = P_{L,R}\psi$, with $P_L = \frac{1}{2}(1-\gamma_5)$, $P_R = \frac{1}{2}(1+\gamma_5)$, one finds

$$\mathcal{L}_{\text{QCD}}^{\text{eff}} = \sum_f \left[\bar{q}_{L,f}i\not{D}q_{L,f} + \bar{q}_{R,f}i\not{D}q_{R,f} - m_f(\bar{q}_{L,f}q_{R,f} + \bar{q}_{R,f}q_{L,f})\right] - \frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu}. \tag{4.111}$$

Therefore, in the absence of a mass term, $\mathcal{L}$ is invariant under the chiral transformations

$$q_L = \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \to V_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = V_L q_L, \tag{4.112}$$

$$q_R \to V_R q_R, \tag{4.113}$$

where V_L and V_R are unitary 3×3 matrices.[5] Instead of m_u, m_d, $m_s \to 0$, it is also useful to consider the two-flavor chiral limit $m_{u,d} \to 0$ and m_s fixed. In this case, the symmetry transformations are

$$\begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \to V_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}, \tag{4.114}$$

and the transformations can be parameterized as

$$V_{L,R} = \exp\left[i\alpha_{L,R} + i\frac{\sigma^a}{2}\alpha^a_{L,R}\right], \tag{4.115}$$

where the Pauli matrices $\frac{\sigma^a}{2}$, $a = 1, 2, 3$, are the generators of $SU(2)$. For the three-flavor case, the generators are the Gell-Mann matrices $\frac{\lambda^a}{2}$, $a = 1, \dots, 8$. We can then consider infinitesimal transformations and Noether's theorem gives a classically conserved current for each transformation $J_\mu \propto \frac{\delta \mathcal{L}}{\delta(\partial^\mu \psi)}\delta\psi$:

$$L_\mu = \bar{q}_L \gamma_\mu q_L, \qquad L^a_\mu = \bar{q}_L \gamma^\mu \frac{\lambda^a}{2} q_L, \tag{4.116}$$

$$R_\mu = \bar{q}_R \gamma_\mu q_R, \qquad R^a_\mu = \bar{q}_R \gamma^\mu \frac{\lambda^a}{2} q_R. \tag{4.117}$$

Instead of left- and right-handed currents, it is convenient to use vector and axial-vector currents:

$$V^\mu = L^\mu + R^\mu = \bar{q}\gamma^\mu q, \tag{4.118}$$

$$A^\mu = R^\mu - L^\mu = \bar{q}\gamma^\mu\gamma^5 q. \tag{4.119}$$

It turns out that A^μ is anomalous, i.e., $\partial_\mu A^\mu \neq 0$ due to quantum effects. More precisely,

$$\partial_\mu A^\mu = \frac{N_c g_s^2}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma} G^{\mu\nu}_a G^{\rho\sigma}_a. \tag{4.120}$$

The remaining $SU(3)_L \times SU(3)_R \times U(1)_V$ transformations are symmetries of the quantum theory. With each current, we can associate a conserved charge

$$Q^a_V = \int d^3x \bar{q}\gamma^0 \frac{\lambda^a}{2} q, \tag{4.121}$$

$$Q^a_A = \int d^3x \bar{q}\gamma^0\gamma^5 \frac{\lambda^a}{2} q. \tag{4.122}$$

The $2 \times 8 + 1$ charges Q_V, Q^a_V, Q^a_A commute with the Hamiltonian $\mathbb{H}_0$ of massless QCD $[Q^a_V, \mathbb{H}_{\text{QCD}}] = [Q^a_A, \mathbb{H}_{\text{QCD}}] = 0$. The question is then whether the spectrum of the theory is symmetric, or whether the symmetry is spontaneously broken. Vafa and Witten have shown that the vector-like symmetries are unbroken $Q^a_V |0\rangle = 0$ [17]. For the axial-vector symmetry, the situation is more complicated. Let us discuss the two possibilities:

1. Unbroken symmetry $Q^a_A |0\rangle = 0$: in this case, the spectrum contains degenerate multiplets of $G = SU(3)_V \times SU(3)_A$.

[5] Note that V_L and V_R are *global* rotations in flavor space, while the gauge transformations act in color space.

2. Spontaneously broken symmetry $Q_A^a|0\rangle \neq 0$: in this case only multiplets of $SU(3)_V \subset G$ appear in the spectrum, and for each broken generator a Goldstone boson arises.

The second case is realized in nature. In particular, a naive derivation of Goldstone's theorem would look as follows. Since $\mathbb{H}Q_A^a|0\rangle = Q_A^a\mathbb{H}|0\rangle = 0$ and $Q_A^a|0\rangle \neq 0$, for each generator there has to be a state with zero energy, so for each broken generator one obtains a massless, parity-odd, spin-0 state. Unfortunately, this simple argument has a flaw:

$$\langle 0| Q_A^a Q_A^b |0\rangle = \int d^3x \int d^3y \, \langle 0| A_0^a(x) A_0^b(y) |0\rangle \tag{4.123}$$

$$= \int d^3x \int d^3y F^{ab}(x-y) = \infty, \tag{4.124}$$

so the "states" $Q_A^a|0\rangle$ have infinite norm. A rigorous proof is obtained by analyzing the correlation function

$$\langle 0| [Q_A^a(t), P^a(t,\mathbf{y})] |0\rangle . \tag{4.125}$$

Inserting a basis of states and using current conservation, one can show that if this matrix element is non-vanishing, then the theory contains a massless particle with the same quantum numbers as $P^a = \bar{q}\frac{\lambda^a}{2}\gamma_5 q$.

The above matrix element can be simplified using the equal time anti-commutation relations

$$\{\psi_{\alpha,r}(t,\mathbf{x}), \psi^\dagger_{\beta,s}(t,\mathbf{y})\} = \delta_{\alpha\beta}\delta_{rs}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{4.126}$$

$$\{\psi,\psi\} = 0, \qquad \{\psi^\dagger,\psi^\dagger\} = 0. \tag{4.127}$$

The commutator has the form

$$[ab, cd] = a\{b,c\}d - ac\{b,d\} + \{a,c\}db - c\{a,d\}b, \tag{4.128}$$

so we get

$$\begin{aligned}[A_0^a(\mathbf{x},0), P^a(\mathbf{y},0)] &= q^\dagger(y)\gamma^5\frac{\lambda^a}{2}\delta^{(3)}(\mathbf{x}-\mathbf{y})\gamma^0\gamma^5\frac{\lambda^a}{2}q(y) - q^\dagger(y)\gamma^0\gamma^5\frac{\lambda^a}{2}\delta^{(3)}(\mathbf{x}-\mathbf{y})\gamma^5\frac{\lambda^a}{2}q(y) \\ &= -2\bar{q}\frac{(\lambda^a)^2}{4}q\delta^{(3)}(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{4.129}$$

Because of $SU(3)_V$ invariance of $|0\rangle$, one can average over the components,[6] leading to

$$\langle 0| [Q_A^a, P^a(x)] |0\rangle = -\frac{1}{8}\sum_b \frac{1}{2}\langle 0| \bar{q}(\lambda^b)^2 q |0\rangle \tag{4.130}$$

$$= -\frac{1}{3}\langle 0| \bar{q}q |0\rangle \tag{4.131}$$

$$= -\frac{1}{3}\langle 0| \bar{u}u + \bar{d}d + \bar{s}s |0\rangle \tag{4.132}$$

$$= -\langle 0| \bar{u}u |0\rangle , \tag{4.133}$$

where we used the fundamental Casimir operator $\sum_a(\lambda^a)^2 = 4C_F\mathbb{1} = \frac{16}{3}\mathbb{1}$. The quark condensate $\bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L$ breaks chiral symmetry.

[6] Formally, this can be derived by considering the analogous commutator $[Q_V^a, S^b]$.

A non-vanishing quark condensate implies that chiral symmetry is spontaneously broken and that there are 8 pseudoscalar Goldstone bosons. Since the quark masses are non-zero, chiral symmetry is not an exact symmetry of QCD. On the other hand, the u-, d-, and s-quark masses are small, so one can treat the mass term of QCD as a perturbation. Looking at the spectrum, one finds that three mesons $\pi^{\pm}$, π^0 are quite light, $M_\pi \approx 140\,\mathrm{MeV}$, and nearly degenerate. Since they also are parity-odd and have spin zero, it is plausible that they are the $SU(2)$ triplet of "Goldstone" bosons associated with the spontaneous breaking of chiral symmetry in the $\begin{pmatrix} u \\ d \end{pmatrix}$ sector:

$$SU(2)_L \times SU(2)_R \to SU(2)_V. \tag{4.134}$$

Since the small mass-term breaks the symmetry explicitly, they acquire a small mass. For this reason, they are called *pseudo* Goldstone bosons. The lowest-lying eight mesons π^+, π^-, π^0, K^+, K^-, K^0, $\bar{K}^0$, and η have $J^P = 0^-$, and so match the pattern of symmetry breaking for $SU(3)_L \times SU(3)_R \to SU(3)_V$. If chiral symmetry were unbroken, one would expect multiplets of the full symmetry group: for each parity-odd meson, there should be a (nearly) degenerate parity-even partner. From these considerations, and from the fact that chiral perturbation theory is very successful in describing the low-energy phenomenology of QCD, one concludes that chiral symmetry is indeed spontaneously broken.

4.4.2. Transformation properties of Goldstone bosons

In order to construct the most general effective Lagrangian, we need to know how the Goldstone-boson fields $\boldsymbol{\pi}$ transform under chiral symmetry. Usually fields transform linearly, as a representation of a group $\boldsymbol{\varphi} \to M(g)\boldsymbol{\varphi}$. For Goldstone bosons, however, the symmetry is realized non-linearly, as we will now see. More details on the CCWZ construction that underlies the following discussion is provided in Appendix C.

Let us consider first the general case of a group G that breaks spontaneously to a subgroup H. There are then $n = n_G - n_H$ Goldstone bosons, which we collect into an n-dimensional vector $\boldsymbol{\pi}(x)$. A realization of the group is a mapping

$$\boldsymbol{\pi} \to \boldsymbol{\pi}' = \boldsymbol{\varphi}(g, \boldsymbol{\pi}) \tag{4.135}$$

for any $g \in G$. This mapping must obey the composition law[7]

$$\boldsymbol{\varphi}(g_1, \boldsymbol{\varphi}(g_2, \boldsymbol{\pi})) = \boldsymbol{\varphi}(g_1 g_2, \boldsymbol{\pi}). \tag{4.136}$$

Remarkably, this property determines $\boldsymbol{\varphi}$ uniquely. To see this, consider the image of the origin $\boldsymbol{\varphi}(g, \boldsymbol{\pi} = 0)$. The elements $h \in H$ map the origin onto itself $\boldsymbol{\varphi}(h, 0) = 0$, since H is linearly realized. Moreover

$$\boldsymbol{\varphi}(gh, 0) = \boldsymbol{\varphi}(g, 0) \quad \forall h \in H, \tag{4.137}$$

so that $\boldsymbol{\varphi}$ is defined on the coset space G/H. It maps an element of G/H into the space of pion fields. Furthermore, it is also invertible since $\boldsymbol{\varphi}(g_1, 0) = \boldsymbol{\varphi}(g_2, 0)$ implies $g_1 H = g_2 H$.[8]

[7] In general $\boldsymbol{\varphi}$ is not a representation, since it is not linear $\boldsymbol{\varphi}(g, \lambda\boldsymbol{\pi}) \neq \lambda\boldsymbol{\varphi}(g, \boldsymbol{\pi})$.

[8] The proof proceeds as in Appendix C:

$$\boldsymbol{\varphi}(e, 0) = 0 = \boldsymbol{\varphi}(g_1^{-1} g_1, 0) = \boldsymbol{\varphi}(g_1^{-1}, \boldsymbol{\varphi}(g_1, 0)) = \boldsymbol{\varphi}(g_1^{-1}, \boldsymbol{\varphi}(g_2, 0)) = \boldsymbol{\varphi}(g_1^{-1} g_2, 0) = 0,$$

and therefore $g_1^{-1} g_2 \in H$, i.e., $g_1 H = g_2 H$.

Accordingly, the function $\boldsymbol{\varphi}(g,0)$ provides a one-to-one mapping between the coset space G/H and the values of the $\boldsymbol{\pi}$ field. The transformation of the field follows from the action of $g \in G$ on the coset space. The only freedom left is the choice of coordinates on G/H.

Let us now consider $G = SU(2)_L \times SU(2)_R = \{(V_L, V_R)|V_L \in SU(2), V_R \in SU(2)\}$ and $H = \{(V,V)|V \in SU(2)\}$. The coset space associated with an element g is the set $\tilde{g}H = \{(\tilde{V}_L V, \tilde{V}_R V)|V \in SU(2)\}$. To parameterize G/H, we select one element of each set $\tilde{g}H$. A possible choice is $U = \tilde{V}_R \tilde{V}_L^\dagger$, since

$$(\tilde{V}_L V, \tilde{V}_R V) = (1, \tilde{V}_R \tilde{V}_L^\dagger)\underbrace{(\tilde{V}_L V, \tilde{V}_L V)}_{\in H}. \tag{4.138}$$

The transformation law of U under G is

$$U \to V_R U V_L^\dagger \tag{4.139}$$

for $g = (V_L, V_R)$. In a final step we need to parameterize $U(x) \in SU(2)$. One can use the standard parameterization

$$U(x) = \exp\left[i\frac{\sigma^a \pi^a}{F}\right] = \exp\left[\frac{i}{F}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\right], \tag{4.140}$$

where we have rewritten the pion field in the linear combinations with definite electric charge. The factor F was introduced to obtain a dimensionless exponent, but it will correspond to the pion decay constant. One could have chosen a different parameterization, e.g., for the $SU(2)$ case often the co-called σ parameterization

$$U(x) = \sqrt{1 - \boldsymbol{\pi}^2/F^2} + \frac{i}{F}\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \tag{4.141}$$

proves beneficial. The π-fields of the two different parameterizations are related by a field redefinition, under which the physics remains unchanged. For $SU(3)$, the standard parameterization is

$$U(x) = \exp\left[\frac{i}{F}\lambda^a \pi^a\right] = \exp\left[\frac{i}{F}\begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}\right]. \tag{4.142}$$

To understand why the field are parameterized in this way, one needs to consider the quark-mass term and the coupling to photons, to which we will turn in the next subsection.

4.4.3. Effective Lagrangian

Now that we know the transformation properties of the Goldstone bosons, it is straightforward to write down the effective Lagrangian in the chiral limit $m_q = 0$. After this we will have to implement the symmetry-breaking terms involving the quark masses.

Under a chiral transformation $U \to V_R U V_L^\dagger$, so we need to find an effective Lagrangian $\mathcal{L}_{\text{eff}}(U)$ that is invariant under this transformation. Since $U(x)$ is dimensionless, the terms with higher orders of $U(x)$ are not suppressed, so instead we order terms by derivatives

$$\mathcal{L}_{\text{eff}} = f_0(U) + f_1(U)\Box U + f_2(U)\partial_\mu U \partial^\mu U + \cdots, \tag{4.143}$$

where for now we have ignored the flavor indices, which will have to be contracted later.

We observe:

1. Chiral symmetry implies $f_0(U) = f_0(V_R U V_L^\dagger)$. Choosing $V_R = \mathbb{1}$, $V_L = U$, this leads to $f_0(U) = f_0(\mathbb{1}) = \text{const}$. Therefore, terms of order $\mathcal{O}(1)$ in the derivative expansion only give an irrelevant constant that can be dropped.

2. The f_1-term can be absorbed into f_2 using integration by parts

$$\int d^4x f_1(U)\Box U = -\int d^4x f_1'(U)\partial_\mu U \partial^\mu U, \tag{4.144}$$

so we can write

$$\mathcal{L}_{\text{eff}} = f(U)\partial_\mu U \partial^\mu U = \tilde{f}(U)\Delta_\mu \Delta^\mu \quad \text{with } \Delta_\mu = (\partial_\mu U)U^\dagger. \tag{4.145}$$

The quantity Δ_μ transforms as $\Delta_\mu \to V_R \Delta_\mu V_R^\dagger$ and is invariant under V_L transformations.

The last question is how the indices of the matrices Δ_μ are contracted. The only possibility to ensure invariance under V_R is

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= C \cdot \text{Tr}[\Delta_\mu \Delta^\mu] = C \cdot \text{Tr}[(\partial_\mu U)U^\dagger(\partial^\mu U)U^\dagger] &(4.146)\\
&= -C \cdot \text{Tr}[\partial_\mu U U^\dagger U \partial^\mu U^\dagger] = -C \cdot \text{Tr}[\partial_\mu U \partial^\mu U^\dagger] &(4.147)\\
&\equiv \frac{F^2}{4}\text{Tr}[\partial_\mu U \partial^\mu U^\dagger], &(4.148)
\end{aligned}$$

where the prefactor has been chosen to get canonically normalized kinetic terms for the pion fields. To see this, we now expand

$$U(x) = \exp\left[\frac{i}{F}\boldsymbol{\pi}\cdot\boldsymbol{\sigma}\right] = \mathbb{1} + \frac{i}{F}\boldsymbol{\pi}\cdot\boldsymbol{\sigma} - \frac{1}{2F^2}\boldsymbol{\pi}^2\mathbb{1} + \mathcal{O}(\pi^3), \tag{4.149}$$

leading to

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\Big(-\frac{1}{F^2}\partial_\mu \pi^a \partial^\mu \pi^b\Big)2\delta^{ab} + \mathcal{O}(\pi^3) = -\frac{1}{2}\partial_\mu \boldsymbol{\pi}\partial^\mu \boldsymbol{\pi} + \mathcal{O}(\pi^3). \tag{4.150}$$

The effective Lagrangian has several remarkable properties:

1. one parameter F determines all π-interactions,

2. symmetry requires interactions with arbitrary many pions,

3. derivative couplings: the interactions vanish if the momenta go to zero.

So far, our effective Lagrangian is only valid in the limit $m_q = 0$ and we should now also implement the quark-mass terms that break the symmetry:

$$\mathcal{L}_M = -\bar{q}_R M q_L - \bar{q}_L M^\dagger q_R, \tag{4.151}$$

with

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{4.152}$$

Note that $\mathcal{L}_M$ would be invariant if M transformed as $M \to V_R M V_L^\dagger$. This property can actually be used to construct $\mathcal{L}_{\text{eff}}(U, M)$: one treats M as an external source that transforms as $M \to V_R M V_L$ (a so-called "spurion" field). $\mathcal{L}_{\rm eff}$ must then be invariant as well. Expanding in M, the lowest invariant term is

$$\mathcal{L}_{\text{symmetry breaking}} = \frac{F^2 B_0}{2} \mathrm{Tr}[MU^\dagger + M^\dagger U]. \tag{4.153}$$

This term gives a mass to the pions. For $SU(2)$ one finds

$$\mathcal{L}_{\text{symmetry breaking}} = \frac{F^2 B_0}{2} \mathrm{Tr}[M]\Big(-\frac{1}{F^2}\boldsymbol{\pi}^2\Big) = -\frac{B_0}{2}(m_u + m_d)\boldsymbol{\pi}^2 \equiv -\frac{M_\pi^2}{2}\boldsymbol{\pi}^2, \tag{4.154}$$

from which we conclude that the masses of the pions are equal and proportional to the sum $m_u + m_d$.

To relate the quantity B_0 to a QCD matrix element, we actually need to treat M as an external source $M \equiv [M(x)]_{ij}$ and then take a functional derivative of the full and effective theory partition function

$$\frac{1}{i}\frac{\delta}{\delta M_{ij}(x)} Z_{\rm QCD} = -\left\langle 0\right| \bar{q}_{L,i}(x) q_{R,j}(x) + \bar{q}_{R,j}(x) q_{L,i}(x) \left|0\right\rangle, \tag{4.155}$$

$$\frac{1}{i}\frac{\delta}{\delta M_{ij}(x)} Z_{\rm eff} = \frac{F^2 B_0}{2} \left\langle 0\right| (U^\dagger)_{ji}(x) + U_{ij}(x) \left|0\right\rangle. \tag{4.156}$$

The classical action is minimized by $\boldsymbol{\pi} = 0$, $U = \mathbb{1}$. Up to pion-loop corrections, we thus have

$$F^2 B_0 \delta_{ij} = -\left\langle 0\right| \bar{q}_{L,i} q_{R,j} + \bar{q}_{R,j} q_{L,i} \left|0\right\rangle, \tag{4.157}$$

$$F^2 B_0 = -\left\langle 0\right| \bar{u}u \left|0\right\rangle = -\left\langle 0\right| \bar{d}d \left|0\right\rangle. \tag{4.158}$$

This shows that B_0 corresponds to the quark condensate in the limit $m_q \to 0$. Taken together with the expansion of the pion mass, we find the relation

$$M_\pi^2 = \underbrace{(m_u + m_d)}_{\text{explicit breaking}} \underbrace{\left(\frac{-\left\langle 0\right| \bar{u}u \left|0\right\rangle}{F^2} \right)}_{\text{spontaneous breaking}} + \mathcal{O}\big(m_q^2\big), \tag{4.159}$$

known as the "Gell-Mann–Oakes–Renner relation" [18]. For $SU(2)$, the three pions have the same mass because the quadratic term in the expansion (4.149) is proportional to the unit matrix, essentially because $\frac{1}{2}\{\sigma^i, \sigma^j\} = \delta^{ij}\mathbb{1}$. In the $SU(3)$ case $\{\lambda^a, \lambda^b\}$ is non-trivial and one finds

$$M_\pi^2 = (m_u + m_d)B_0 + \mathcal{O}\big(m_q^2\big), \tag{4.160}$$

$$M_{K^\pm}^2 = (m_u + m_s)B_0 + \mathcal{O}\big(m_q^2\big), \tag{4.161}$$

$$M_{K^0,\bar{K}^0}^2 = (m_d + m_s)B_0 + \mathcal{O}\big(m_q^2\big), \tag{4.162}$$

$$M_\eta^2 = \frac{1}{3}(m_u + m_d + 4m_s)B_0 + \mathcal{O}\big(m_q^2\big). \tag{4.163}$$

This explains why $M_K^2 \gg M_\pi^2$, because $m_s \gg m_u, m_d$. In addition, one has the Gell-Mann–Okubo mass formula [19, 20]

$$M_\pi^2 - 4M_K^2 + 3M_\eta^2 = \mathcal{O}\big(m_q^2\big). \tag{4.164}$$

To understand how the mesons interact with photons, W-, and Z-bosons, it is useful to introduce external sources with the appropriate quantum numbers both in the full and the effective theory. For QCD, we add

$$\mathcal{L}_{\mathrm{QCD}} = \mathcal{L}_0 + \mathcal{L}_1, \tag{4.165}$$

$$\mathcal{L}_0 = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \bar{q}i\not{D}q, \tag{4.166}$$

$$\mathcal{L}_1 = v^a_\mu V^\mu_a + a^a_\mu A^\mu_a - s^a S_a - p^a P_a, \tag{4.167}$$

with sources

$$V^\mu_a = \bar{q}\gamma^\mu\frac{\lambda_q}{2}q, \qquad A^\mu_a = \bar{q}\gamma^\mu\gamma_5\frac{\lambda_q}{2}q, \qquad S_a = \bar{q}\frac{\lambda_q}{2}q, \qquad P_a = \bar{q}i\gamma_5\frac{\lambda_q}{2}q, \tag{4.168}$$

and one can also include singlet currents via $\lambda_0 = \sqrt{\frac{2}{3}}\mathbb{1}$. The external fields $v^a_\mu(x)$, $a^a_\mu(x)$, $s^a(x)$, $p^a(x)$ can be used to probe different aspects of QCD, e.g., quark masses are included in $s^a(x)$. To construct $\mathcal{L}_{\mathrm{eff}}$ in the presence of these sources, one can use the fact that $\mathcal{L}_{\mathrm{QCD}}$ becomes invariant under *local* transformations

$$q_L(x) \to V_L(x)q_L(x), \qquad q_R(x) \to V_R(x)q_R(x), \tag{4.169}$$

provided the external fields transform like gauge fields:

$$r_\mu = v_\mu + a_\mu \to V_R(v_\mu + a_\mu)V_R^\dagger - i(\partial_\mu V_R)V_R^\dagger, \tag{4.170}$$

$$l_\mu = v_\mu - a_\mu \to V_L(v_\mu - a_\mu)V_L^\dagger - i(\partial_\mu V_L)V_L^\dagger, \tag{4.171}$$

$$s + ip \to V_R(s+ip)V_L^\dagger, \tag{4.172}$$

where $v_\mu = v^a_\mu\frac{\lambda^a}{2}$, etc., and the task is then to construct a locally invariant effective Lagrangian. At leading order, it is sufficient to replace ∂_μ by the covariant derivative:

$$iD_\mu U = i\partial_\mu U + (v_\mu + a_\mu)U - U(v_\mu - a_\mu), \tag{4.173}$$

where v_μ, a_μ count as $\mathcal{O}(p)$, so that

$$\mathcal{L}_{\mathrm{eff}} = \frac{F^2}{4}\mathrm{Tr}[D_\mu U D^\mu U^\dagger] + \frac{F^2 B_0}{2}\mathrm{Tr}[\chi U^\dagger + \chi^\dagger U] + \mathcal{O}(p^4), \tag{4.174}$$

with $\chi = s + ip$ and the convention $D_\mu U^\dagger \equiv (D_\mu U)^\dagger$. We are now in the position to interpret the second free parameter F, by considering the axial-vector current, i.e.,

$$D_\mu U = \frac{i}{F}\boldsymbol{\sigma}\cdot\partial_\mu\boldsymbol{\pi} - 2i\boldsymbol{a}_\mu\cdot\frac{\boldsymbol{\sigma}}{2} + \mathcal{O}(\boldsymbol{\pi}^2). \tag{4.175}$$

For the coupling of a single axial-vector current we thus find

$$\frac{F^2}{4}\frac{4}{F}\Big(-\boldsymbol{a}_\mu\cdot\partial^\mu\boldsymbol{\pi}\Big) = -F\boldsymbol{a}_\mu\cdot\partial^\mu\boldsymbol{\pi}, \tag{4.176}$$

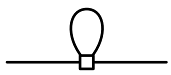
and matching to Eq. (4.75) proves $F = F_\pi$ at leading order.

At $\mathcal{O}(p^4)$ $\mathcal{L}_{\text{eff}}$ has the form [2,3]

$$\begin{aligned}\mathcal{L}^{(4)} &= \frac{l_1}{4}(\mathrm{Tr}[D_\mu U D^\mu U^\dagger])^2 + \frac{l_2}{4}\mathrm{Tr}[D_\mu U D_\nu U^\dagger] \times \mathrm{Tr}[D^\mu U D^\nu U^\dagger] \\ &+ \frac{l_3}{4}(\mathrm{Tr}[\chi U^\dagger + U\chi^\dagger])^2 + \frac{l_4}{4}\mathrm{Tr}[D_\mu \chi D^\mu U^\dagger + D_\mu U D^\mu \chi^\dagger] + \cdots \end{aligned} \tag{4.177}$$

For $SU(3)$ $\mathcal{L}^{(4)}$ has 12 coupling constants, while for $SU(2)$ 10 such low-energy constants arise.

To perform calculations beyond leading order, one needs one-loop graphs from $\mathcal{L}^{(2)}$, which also count as $\mathcal{O}(p^4)$, e.g.,


$\mathcal{L}^{(2)}$

$$\propto \int d^4k \frac{1}{k^2 - M_\pi^2} k^2 \propto M_\pi^4 \tag{4.178}$$

contributes to the $\mathcal{O}(m_q^2)$ corrections in the mass formulae (4.160). In particular, loop contributions are suppressed by a loop factor $1/(16\pi^2)$ and, for dimensional reasons, by $1/F_\pi^2$, leading to an expansion in $1/(4\pi F_\pi)^2$.[9] From the loop suppression one thus expects an expansion in

$$\left\{ \frac{p^2}{\Lambda_\chi^2}, \frac{M_\pi^2}{\Lambda_\chi^2}, \frac{M_K^2}{\Lambda_\chi^2} \right\}, \tag{4.179}$$

where M_π and M_K apply to the $SU(2)$ and $SU(3)$ expansions, respectively. The RG estimate for the scale of chiral symmetry breaking can be compared to the first resonance in the spectrum, which gives rise to

$$0.775\,\mathrm{GeV} \approx M_\rho \lesssim \Lambda_\chi \lesssim 4\pi F_\pi \approx 1.2\,\mathrm{GeV}, \tag{4.180}$$

and thus a scale around 1 GeV. In practice, the expansion proceeds not just in terms of powers of Eq. (4.179), but the loop corrections can generate logarithmic dependences on momenta and masses.

Finally, we mention a complication that appears in the construction of chiral Lagrangians, which is related to the Wess–Zumino–Witten (WZW) anomaly [22,23]. The point is that $\mathcal{L}_{\text{QCD}}$ and $\mathcal{L}_{\text{eff}}$ are invariant under local chiral transformations, but the partition function

$$Z[\nu, a, s, p] = \int \mathcal{D}q\,\mathcal{D}\bar{q}\,\mathcal{D}A_\mu\, e^{i\int d^4x(\mathcal{L}_0 + \mathcal{L}_1)} = e^{iS_{\text{eff}}[\nu,a,s,p]} \tag{4.181}$$

is not invariant if the external sources are non-zero, because of anomalies in the fermion determinant. Since the effective theory does not involve fermion fields, invariance of $\mathcal{L}_{\text{eff}}$ leads to invariance of the partition function. To correct for this mismatch, one needs to add $\mathcal{L}_{\text{eff}}$ a term that reproduces the charge of the QCD partition function. This term is called the WZW term $\mathcal{L}_{\text{WZW}}$. The full effective theory Lagrangian is then

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{WZW}}. \tag{4.182}$$

[9] This argument can be made more rigorous in terms of the renormalization group [21].

The WZW term is $\mathcal{O}(p^4)$ and does not involve any low-energy constants. In contrast to $\mathcal{L}_{\text{inv}}$, the terms in $\mathcal{L}_{\text{WZW}}$ contain odd numbers of Goldstone-boson fields. In particular, it contains a term describing an interaction of two vector fields with a π^0, which leads to

$$\Gamma_{\pi^0\to\gamma\gamma} = \frac{\alpha^2 N_c^2 M_{\pi^0}^3}{64\pi^3 F_\pi^2}\left(Q_u^2 - Q_d^2\right)^2 = 7.749(15)\,\text{eV}. \tag{4.183}$$

The good agreement with the experimental value $7.802(117)\,\text{eV}$ [24] is sometimes sold as evidence for $N_c = 3$. However, Bär and Wiese pointed out that $Q_u^2 - Q_d^2 = \frac{1}{N_c}$ for N_c colors (to ensure anomaly cancellation in the SM), so that the rate does not depend on N_c [25]. The π^0 decay is much faster than the one of the charged pion because it is mediated by strong/electromagnetic instead of electroweak interactions.

4.5. The Standard Model as an EFT

We have now covered (almost[10]) all sectors of the SM and discussed the corresponding EFTs. In particular, we have worked our way up in energy and have integrated out heavy leptons, quarks, and gauge bosons. In each case we wrote down the relevant operators up to $d = 6$. In all cases, the main motivation for the EFT approach was either to simplify calculations in the SM or, as for strong interactions at low energies, to actually make analytic calculations possible.

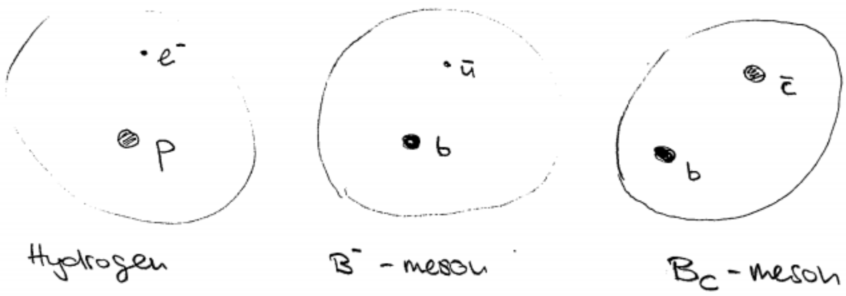
Going one step further, one could also consider the entire SM as an EFT, i.e., assume that all potential new particle arise above the scale of electroweak symmetry breaking and can thus be described by effective operators that obey the SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. While there is only a single operator at $d = 5$ (related to neutrino Majorana masses), there are many possibilities at $d = 6$, and finding the minimal set is non-trivial. A complete set was first written down by Buchmüller and Wyler [26]. However, the *minimal* set was only constructed in 2010 [27], comprising $15 + 19 + 25 = 59$ different operators (bosonic, two-fermion, and four-fermion, respectively). If baryon number is violated, four additional operators appear.

[10]Heavy-quark EFT requires a non-relativistic formalism, see Chapter 5.

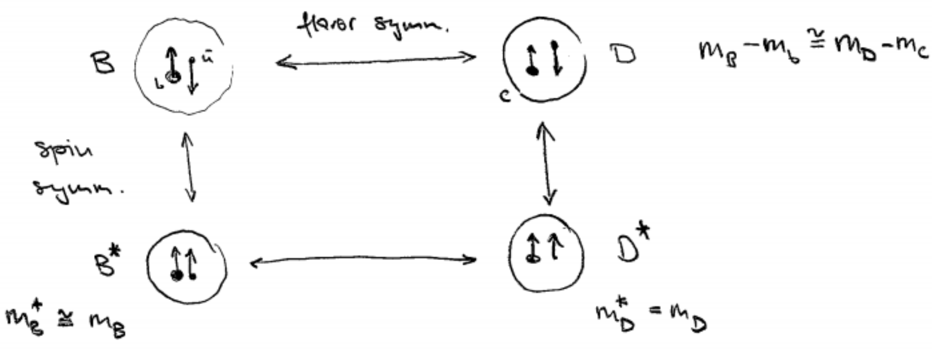
5. Non-relativistic effective theories

We have considered several EFTs that are obtained by integrating out heavy particles. However, in many cases heavy particles are present even at very low energy. The reason are conservation laws for particle numbers such as lepton number conservation ($L = L_{e^-} - L_{e^+}$) and baryon (or quark) number conservation. If we neglect the weak interaction, then each lepton and quark flavor is separately conserved.

The proper framework to describe heavy particles at low momentum are non-relativistic EFTs. Examples of systems that can be studied with such techniques are atoms, mesons with heavy (i.e., bottom or charm) quarks, and protons interacting with slow pions, etc.



A heavy B-meson has similarities to a hydrogen atom, but an important difference is that the light degrees of freedom inside the B-meson are still highly relativistic and strongly interacting. Nevertheless some properties of hydrogen carry over: the energy of the B-meson is to good accuracy independent of the b-quark spin. Also, the energy spectrum of B-mesons is independent of the heavy-quark mass to good approximation:



Heavy quark effective theory (HQET) will allow us to derive such relations in the limit $m_Q \to \infty$ and to systematically analyze the $\frac{1}{m_Q}$ corrections. For systems such as hydrogen or a B_c meson also lighter fermions (e^- or $\bar{c}$ respectively) can be treated non-relativistically. The EFT for this case is non-relativistic QED/QCD (NRQED/NRQCD). HQET and NRQCD have the same Lagrangian but different power counting.

5.1. Heavy-quark effective theory

Interactions of the heavy quark Q with the light constituents of a heavy-to-light meson will change its momentum by amounts of order $\Lambda_{\text{QCD}} \sim 1$ GeV, but its velocity is barely changed $\delta v_Q^\mu = \frac{\Delta p_Q^\mu}{m_Q} \ll 1$. To analyze such systems, we introduce a reference vector v^μ, $v^2 = 1$, in the direction of the heavy quark and split

$$p_Q^\mu = m_Q v^\mu + r^\mu, \tag{5.1}$$

so that the residual momentum r^μ is $\mathcal{O}(\Lambda_{\text{QCD}})$. A popular choice for v^μ is the meson velocity $v^\mu = \frac{P_M^\mu}{m_M}$. The EFT then corresponds to an expansion in the residual momentum r^μ over the heavy quark mass m_Q.

On the level of the quark field the decomposition of the momentum is achieved by splitting off the large phase $e^{-im_Q v\cdot x}$ from the field:

$$\psi_Q(x) = e^{-im_Q v\cdot x}\{h_v(x) + H_v(x)\}, \tag{5.2}$$

where

$$h_v(x) = e^{im_Q v\cdot x} P_+ \psi_Q(x), \qquad P_+ = \frac{\mathbb{1} + \not{v}}{2}, \tag{5.3}$$

$$H_v(x) = e^{im_Q v\cdot x} P_- \psi_Q(x), \qquad P_- = \frac{\mathbb{1} - \not{v}}{2}. \tag{5.4}$$

The projection operators P_+ and P_- split the field into the large ("upper") components $h_v(x)$ and the small ("lower") components $H_v(x)$. They obey

$$\not{v} h_v(x) = h_v(x), \qquad \not{v} H_v(x) = -H_v(x). \tag{5.5}$$

Let us insert this decomposition into the Dirac Lagrangian:

$$\mathcal{L}_Q = \bar{\psi}_Q (i\not{D} - m_Q)\psi_Q \tag{5.6}$$

$$= \bar{h}_v i\not{D} h_v + \bar{H}_v (i\not{D} - 2m_Q) H_v + \bar{H}_v i\not{D} h_v + \bar{h}_v i\not{D} H_v, \tag{5.7}$$

which further simplifies to

$$\mathcal{L}_Q = \bar{h}_v iv\cdot D h_v + \bar{H}_v(-iv\cdot D - 2m_Q)H_v + \bar{H}_v i\not{D}_\perp h_v + \bar{h}_v i\not{D}_\perp H_v, \tag{5.8}$$

when expressed in terms of the component $D_\perp^\mu = D^\mu - v\cdot D v^\mu$ perpendicular to v^μ. Here we used that

$$\bar{h}_v \gamma^\mu h_v = \bar{h}_v \gamma^\mu \not{v} h_v = -\bar{h}_v \gamma^\mu h_v + 2v^\mu \bar{h}_v h_v = v^\mu \bar{h}_v h_v, \tag{5.9}$$

$$\bar{H}_v \gamma^\mu H_v = -v^\mu \bar{H}_v H_v, \tag{5.10}$$

$$\bar{H}_v \not{v} h_v = \bar{h}_v \not{v} H_v = 0. \tag{5.11}$$

The EOM for H_v is

$$(-iv \cdot D - 2m_Q)H_v + i\not{D}_\perp h_v = 0, \tag{5.12}$$

which can be formally inverted as

$$H_v = \frac{1}{2m_Q} \sum_{n=0}^{\infty} \left(-\frac{iv \cdot D}{2m_Q} \right)^n i\not{D}_\perp h_v. \tag{5.13}$$

This shows that H_v is suppressed with respect to h_v by a factor $\frac{r_\perp}{2m_Q}$ and can be integrated out. Since the action is quadratic in the fields, this step can even be done exactly. At the classical level, the result is obtained by inserting the solution of the EOM for H_v back into $\mathcal{L}_Q$. At leading order one obtains

$$\mathcal{L}_Q = \bar{h}_v iv \cdot D h_v + \frac{1}{2m_Q} \underbrace{\bar{h}_v i\not{D}_\perp i\not{D}_\perp h_v}_{\text{power corrections}} + \mathcal{O}\Big(\frac{1}{m_Q^2}\Big). \tag{5.14}$$

The power corrections can be further rewritten as

$$\begin{aligned} i\not{D}_\perp i\not{D}_\perp &= iD_\mu^\perp iD_\nu^\perp (\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]) && (5.15)\\ &= iD_\mu^\perp iD_\nu^\perp (g_{\mu\nu} - i\sigma^{\mu\nu}) && (5.16)\\ &= (iD_\perp)^2 + \frac{i}{2}[D_\mu^\perp, D_\nu^\perp]\sigma^{\mu\nu} && (5.17)\\ &= (iD_\perp)^2 + \frac{g_s}{2}\sigma^{\mu\nu}G_{\mu\nu}, && (5.18) \end{aligned}$$

where we used the analog of Eq. (4.3) for the QCD field strength tensor.[1] The resulting Lagrangian

$$\mathcal{L}_Q = \bar{h}_v iv \cdot D h_v + \frac{1}{2m_Q}\bar{h}_v (iD_\perp)^2 h_v + \frac{g_s}{4m_Q}\bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v \tag{5.19}$$

simplifies further when going to the rest frame $v^\mu = (1, \mathbf{0})$

$$\mathcal{L}_Q = \bar{h}_v iD_t h_v + \frac{1}{2m_Q}\bar{h}_v \boldsymbol{D}^2 h_v - \frac{g_s}{2m_Q}\bar{h}_v \boldsymbol{\sigma} \cdot \mathbf{B}_c h_v, \tag{5.20}$$

where $\mathbf{B}_c$ is the chromomagnetic field and the notation has been changed to two-component spinors. The first term is independent of the quark spin ("heavy-quark spin symmetry") and the quark mass ("heavy-quark flavor symmetry"). The second term breaks heavy-quark flavor symmetry but maintains the spin symmetry, while the third term violates both.

5.2. Connection to quantum mechanics

Let us go into the rest frame of the heavy quark $v^\mu = (1, \mathbf{0})$. The projection operator is then

$$P_+ = \frac{1}{2}(\mathbb{1} + \gamma_0) = \begin{pmatrix} \mathbb{1} & \\ & 0 \end{pmatrix}, \tag{5.21}$$

[1]The corrections from $D_\mu^\perp$ vs. D_μ are symmetric in $\mu \leftrightarrow \nu$ and thus vanish upon contraction with $\sigma^{\mu\nu}$.

i.e., P_+ projects out the upper two components of the Dirac field. When considering QED instead of QCD,

$$\psi_Q \to \psi_e, \qquad iD_\mu \to i\partial_\mu - eA_\mu,$$

the magnetic operator becomes

$$-\frac{e}{4m_e}\bar{h}_v \sigma^{\mu\nu} F_{\mu\nu} h_v = -\frac{ie}{8m_e}\bar{h}_v \underbrace{(\sigma^i\sigma^j - \sigma^j\sigma^i)}_{2i\varepsilon^{ijk}\sigma^k} \underbrace{(\partial_i A_j - \partial_j A_i)}_{2\partial_i A_j} h_v \tag{5.22}$$

$$= \frac{e}{2m_e}\bar{h}_v \boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\mathbf{A})h_v = \frac{e}{2m_e}\bar{h}_v\boldsymbol{\sigma}\cdot\mathbf{B}h_v. \tag{5.23}$$

The effective Lagrangian for a slow electron (described by a field X) is therefore

$$\mathcal{L} = \bar{X} iD_t X - \bar{X}\frac{(i\boldsymbol{D})^2}{2m_e}X + \frac{e}{2m_e}\bar{X}\boldsymbol{\sigma}\cdot\mathbf{B}X. \tag{5.24}$$

The EOM associated with this Lagrangian is the Schrödinger equation for an e^- interacting with a photon field. The free propagator associated with $\mathcal{L}$

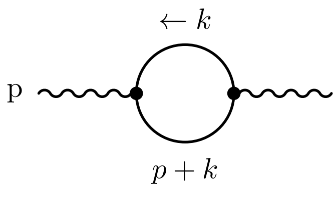
$$\triangle_x = \frac{1}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon} \tag{5.25}$$

has only a *single* pole, in contrast to a relativistic propagator:

$$\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{2\omega}\left[\underbrace{\frac{1}{p^0 - \omega + i\varepsilon}}_{\text{particle}} - \underbrace{\frac{1}{p^0 + \omega - i\varepsilon}}_{\text{antiparticle}}\right] \tag{5.26}$$

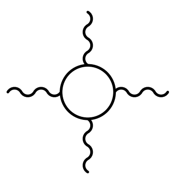
$$= \frac{1}{2m}\frac{1}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon} + \ldots, \tag{5.27}$$

where $\omega = m + \frac{\mathbf{p}^2}{2m} + \ldots$, $p^0 = m + E$. This has important consequences: since the theory no longer contains anti-particles, closed fermion loops vanish:



$$\propto \int d^dk \frac{1}{(E + k^0) - \frac{(\mathbf{p}+\mathbf{k})^2}{2m} + i\epsilon}\frac{1}{k^0 - \frac{\mathbf{k}^2}{2m} + i\epsilon} = 0, \tag{5.28}$$

because we can choose the k^0 integration contour without encountering a pole (since $\operatorname{Im} k^0 < 0$ for all poles). Therefore all fermion loops vanish in HQET. The effect of virtual anti-particles can be absorbed into the Wilson coefficients of the operators in $\mathcal{L}_{\text{eff}}$, e.g.,



is represented by the Euler–Heisenberg terms in $\mathcal{L}_{\text{eff}}$.

Despite this observation, our theory is not simply quantum mechanics, since the electromagnetic field is a fully relativistic quantum field. To obtain QM, we need to treat also the electromagnetic field as a classical one. Let us therefore assume that $A^\mu = \phi(0, \mathbf{x})$ is a fixed classical potential (e.g., the Coulomb field of a proton).

Now the field operator fulfills

$$i\partial_t \hat{X} = [\hat{X}, \hat{\mathbb{H}}] = \left(-\frac{\boldsymbol{\nabla}^2}{2m} + e\phi(\mathbf{x}) \right) \hat{X}. \tag{5.29}$$

The solutions of the time-independent Schrödinger equation

$$\left(-\frac{\boldsymbol{\nabla}^2}{2m} + e\phi(\mathbf{x}) \right) \varphi_n(x) = E_n \varphi_n(x) \tag{5.30}$$

form a complete set of functions, which can be used to expand

$$\hat{X}(t, \mathbf{x}) = \sum_i \hat{a}_i e^{-iE_i t} \varphi_i(x). \tag{5.31}$$

The operator $\hat{a}_i$ annihilates the state with associated wave function $\varphi_i(x)$. Now the system is indeed quantum mechanical: the one-particle states are $|i\rangle = a_j^\dagger |0\rangle$ and they have associated wave functions

$$\langle 0| \hat{X}(t, \mathbf{x}) |i\rangle = e^{-iE_i t} \varphi_i(x) \tag{5.32}$$

that fulfill the Schrödinger equation.

To summarize

1. The EFT for a non-relativistic particle has a Lagrangian that has the Schrödinger equation as EOM.

2. There are no anti-particles in the EFT, their effect can be absorbed into the Wilson coefficients, since they are highly virtual.

3. Treating the photon as a classical background field we recover quantum mechanics.

5.3. Non-relativistic QED and QCD

Let us finally consider the effective theory relevant for the description of bound states of two heavy particles, e.g., positronium (e^+e^-), muonium ($\mu^+\mu^-$), bottomonium ($\bar{b}b$), and charmonium ($\bar{c}c$). The effective theories are called NRQED and NRQCD, respectively, and are closely related to HQET, except for the fact that we now deal with both a particle, described by a two-component spinor ψ, and an anti-particle, which we denote by χ. The effective Lagrangian has the form

$$\mathcal{L}_{\text{NR}} = \mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{\text{mixed}} + \mathcal{L}_{\text{light}}. \tag{5.33}$$

$\mathcal{L}_{\text{mixed}}$ contains operators involving both χ and ψ fields. $\mathcal{L}_{\text{light}}$ is the QCD Lagrangian for the light quarks plus higher-dimensional operators. The Lagrangians for the ψ field is nothing but the HQET Lagrangian evaluated for $v^\mu = (1, \mathbf{0})$, since it is natural to work in the rest frame of the bound state. Therefore,

$$\begin{aligned}\mathcal{L}_\psi &= \psi^\dagger\Big(iD_t + \frac{\boldsymbol{D}^2}{2m_\psi}\Big)\psi + \frac{1}{8m_\psi^3}\psi^\dagger\boldsymbol{D}^4\psi + \frac{gC_1}{2m_\psi}\psi^\dagger\boldsymbol{\sigma}\cdot\mathbf{B}\psi + \frac{gC_2}{8m_\psi^2}\psi^\dagger\big(\boldsymbol{D}\cdot\mathbf{E} - \mathbf{E}\cdot\boldsymbol{D}\big)\psi \\ &\quad + \frac{gC_3}{8m_\psi^2}\psi^\dagger\big(i\boldsymbol{D}\times\mathbf{E} - \mathbf{E}\times i\boldsymbol{D}\big)\cdot\boldsymbol{\sigma}\psi,\end{aligned} \tag{5.34}$$

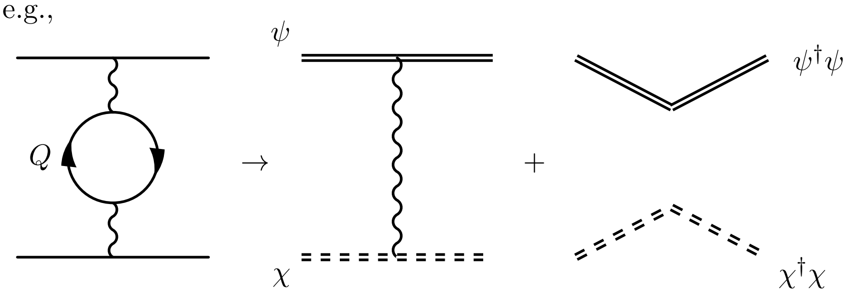
where g is the gauge coupling, $g = g_s$ or $g = -e$. Moreover, we have included $1/m_Q^2$- and $1/m_Q^3$-suppressed terms because the power counting is different than in HQET. Next, $\mathcal{L}_\chi$ is obtained as the charge conjugate of $\mathcal{L}_\psi$,

$$\mathcal{L}_\chi = \mathcal{L}_\psi\big|_{\psi\to\chi,\, A^\mu\to -A^\mu}, \tag{5.35}$$

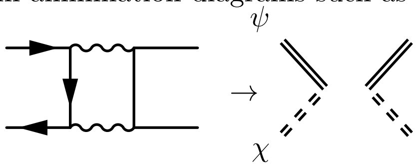
and the lowest-dimensional operators in $\mathcal{L}_{\text{mixed}}$ are four-quark operators, e.g.,

$$\mathcal{L}_{\text{mixed}} = \frac{C_4}{m^2}\psi^\dagger\psi\,\chi^\dagger\chi + \frac{C_5}{m^2}\psi^\dagger\boldsymbol{\sigma}\sigma_2\chi\,\chi^\dagger\sigma_2\boldsymbol{\sigma}\psi. \tag{5.36}$$

The first operator arises when high-energy contributions to the scattering of χ and ψ are integrated out, e.g.,



The second operator arises in annihilation diagrams such as



These diagrams only exist if χ is the anti-particle of ψ. They have an imaginary part, which describes the decay $\psi\chi \to \gamma\gamma$ (or gg), and accordingly C_5 is imaginary. The effective $\mathbb{H}$ is not Hermitian and the theory is not unitary! However, there is a good physical reason for this violation of unitarity: bound states, such as e^+e^-, decay over time. The imaginary part of $\mathbb{H}$ encodes the decay rate. The probability for finding the e^- in e^+e^- is not 1 for all times, because it will annihilate sooner or later.

This is the first complication compared to HQET. The second one is that the static Lagrangian

$$\mathcal{L} = \psi^\dagger i\partial_t\psi \tag{5.37}$$

cannot serve as a starting point in non-relativistic theories. There are formal arguments to show this, but the simple physical reason is that the e^+e^- in the bound state are not static. They are close to their mass shell $E = \frac{\mathbf{p}^2}{2m} + \dots$ and we thus should count $D_t \sim \frac{\boldsymbol{D}^2}{2m} \sim \frac{m\mathbf{v}^2}{2}$ as of the same order. Instead of powers of $1/m_Q$, we should count powers of $v = |\mathbf{v}|$. A third complication is that the multiple photon/gluon exchanges between ψ and χ are unsuppressed. More precisely, the exchange of Coulomb gluons needs to be taken into account to all orders. In Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, the gauge Lagrangian reads

$$\begin{aligned} -\frac{1}{4}G^{\mu\nu}G_{\mu\nu} &= \frac{1}{2}G^{0i}G_{0i}-\frac{1}{4}G^{ij}G_{ij} \\ &= \frac{1}{2}\left[(\partial_i A_0)^2+(\partial_0 A_i)^2-(\partial_i A_j)^2+\text{“non-Abelian” terms}\right]. \end{aligned} \tag{5.38}$$

The field A_0 has no time derivatives and is thus not propagating. Since the action is quadratic, one can integrate out A^0. Its effect is then described by a potential, which is just the Fourier transform of its propagator

$$V(\mathbf{x}-\mathbf{y}) = g_s^2\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{1}{\mathbf{k}^2}=\frac{g_s^2}{4\pi|\mathbf{x}-\mathbf{y}|}. \tag{5.39}$$

The leading-order effective Lagrangian for a non-relativistic particle–antiparticle pair is then

$$\mathcal{L}_{\text{NR}} = \int d^3x\,\psi^\dagger\left[i\partial_t+\frac{\boldsymbol{\nabla}^2}{2m}\right]\psi-\int d^3x_1\int d^3x_2\psi^\dagger(x_1)t^a\psi(x_1)\chi^\dagger(x_2)t^a\chi(x_2)V(\mathbf{x}_1-\mathbf{x}_2). \tag{5.40}$$

Accounting for $V(\mathbf{x})$ to all orders amounts to solving the Schrödinger equation. The remaining terms are treated as perturbations. Unfortunately, we thus found that the problem involves three different scales

	m (hard)	mv (soft)	mv^2 (ultrasoft)
e^+e^- value	0.5 MeV	3.7 keV	25 eV

where we have used that for positronium $V\sim\alpha$. These three scales make it difficult to organize the computations. In particular, in dimensional regularization the non-relativistic integrals receive contributions from the hard region, since the scale m_Q appears in the integrand. Initially people used to perform the computations with a hard cutoff, which avoids this problem but makes computations extremely cumbersome. Using the threshold expansion [28], which is also called the “strategy of regions,” it became possible to eliminate the unwanted hard corrections in dimensional regularization and to separate the soft and ultrasoft corrections. An EFT approach that implements this separation on the level of the Lagrangian is “velocity NRQCD” or “vNRQCD,” first proposed in Ref. [29]. An earlier solution called “potential NRQCD” (“pNRQCD”) [30] amounts to integrating out the soft scale mv and to constructing an effective theory containing only ultrasoft degrees of freedom:

$$\begin{array}{ccc} \mathcal{L}_{\text{QCD}} & (\mu>m) & \text{hard + soft + ultrasoft} \\ \downarrow & & \\ \mathcal{L}_{\text{NRQCD}} & (m>\mu>mv) & \text{soft + ultrasoft} \\ \downarrow & & \\ \mathcal{L}_{\text{pNRQCD}} & (mv>\mu>mv^2) & \text{ultrasoft} \end{array}$$

The fields in pNRQCD are not quarks and anti-quarks, but color-singlet and color-octet $\bar{Q}Q$-pairs:

$$S\equiv S(r)\sim\chi^\dagger(-\frac{r}{2})\psi(\frac{r}{2}),\quad O^a\sim\chi^\dagger t^a\psi, \tag{5.41}$$

where S is the singlet and O the octet pair, which interact through potentials $V(r)$ and ultrasoft gluons.

A. Loop integrals in dimensional regularization

In this appendix we derive the standard formula

$$I(\alpha,\beta,\Delta) = \int \frac{d^dk}{(2\pi)^d}\frac{(k^2)^\alpha}{(k^2-\Delta+i\epsilon)^\beta} = \frac{i(-1)^{\alpha+\beta}}{(4\pi)^{d/2}}\frac{\Gamma\Big(\alpha+\frac{d}{2}\Big)}{\Gamma\Big(\frac{d}{2}\Big)}\frac{\Gamma\Big(\beta-\alpha-\frac{d}{2}\Big)}{\Gamma(\beta)}\Delta^{\alpha-\beta+d/2}, \quad \text{(A.1)}$$

and show that all one-loop integrals can be brought into this form.

As a first step, we consider $I(\alpha,\beta,\Delta)$ for parameters $\alpha+d/2>0$ (IR convergence) and $\beta-\alpha-d/2>0$ (UV convergence), by performing a Wick rotation to Euclidean space $k^0 = ik_E^0$, $\mathbf{k} = \mathbf{k}_E$. Dropping the Euclidean label in the following, this gives

$$\begin{aligned} I(\alpha,\beta,\Delta) = i(-1)^{\alpha+\beta}\int \frac{d^dk}{(2\pi)^d}\frac{(k^2)^\alpha}{(k^2+\Delta)^\beta} &= \frac{i(-1)^{\alpha+\beta}\Omega_d}{(2\pi)^d}\int_0^\infty dk\frac{k^{d-1+2\alpha}}{(k^2+\Delta)^\beta} \\ &= \frac{i(-1)^{\alpha+\beta}\Omega_d}{2(2\pi)^d}\Delta^{\alpha-\beta+d/2}\int_0^\infty dx\frac{x^{\alpha-1+d/2}}{(1+x)^\beta}, \end{aligned} \quad \text{(A.2)}$$

where we used polar coordinates in d dimensions with area Ω_d of the unit sphere und changed the integration to $x = k^2/\Delta$. The remaining integral can be brought into standard form by the transformation $y = x/(1+x)$, with $dx = dy/(1-y)^2$,

$$\begin{aligned} I(\alpha,\beta,\Delta) &= \frac{i(-1)^{\alpha+\beta}\Omega_d}{2(2\pi)^d}\Delta^{\alpha-\beta+d/2}\int_0^1 dy\, y^{\alpha+d/2-1}(1-y)^{\beta-\alpha-d/2-1} \\ &= \frac{i(-1)^{\alpha+\beta}\Omega_d}{2(2\pi)^d}\Delta^{\alpha-\beta+d/2}\frac{\Gamma\Big(\alpha+\frac{d}{2}\Big)\Gamma\Big(\beta-\alpha-\frac{d}{2}\Big)}{\Gamma(\beta)}, \end{aligned} \quad \text{(A.3)}$$

where in the last step we applied the general relation for the Beta function

$$\mathrm{B}(a,b) = \int_0^1 dy\, y^{a-1}(1-y)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad \text{(A.4)}$$

The final result (A.1) then follows with $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$, which can be derived from Gaussian integrals in d dimensions

$$\begin{aligned} &\int d^dx\, e^{-\mathbf{x}^2} = \left[\int dx\, e^{-x^2}\right]^d = \pi^{d/2} \\ &= \Omega_d\int_0^\infty dx\, x^{d-1}e^{-x^2} = \frac{\Omega_d}{2}\int_0^\infty dy\, y^{d/2-1}e^{-y} = \frac{\Omega_d}{2}\Gamma\Big(\frac{d}{2}\Big). \end{aligned} \quad \text{(A.5)}$$

The derivation only applies for integer values of d, with the general result *defined* by

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma\Big(\frac{d}{2}\Big)}. \quad \text{(A.6)}$$

Similarly, our derivation of (A.1) only applies as long as the integral is IR and UV convergent. However, the right-hand side is an analytic function of d, α, and β except for poles for $\alpha + d/2$ and $\beta - \alpha - d/2$ at $0, -1, -2, \ldots$. The integral is then defined by the analytic continuation in these variables. To evaluate the limit of $d = 4 - 2\epsilon \to 4$, one often needs the expansion

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\Big(\frac{1}{\epsilon} - \gamma_E + 1 + \cdots + \frac{1}{n}\Big) + \mathcal{O}(\epsilon), \tag{A.7}$$

where $\gamma_E = 0.5772\ldots$ is the Euler–Mascheroni constant.

To demonstrate that all one-loop diagrams indeed take the form (A.1) one uses Feynman parameterizations to combine multiple propagators into a single one, in the simplest case using

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}. \tag{A.8}$$

A non-trivial example including Lorentz indices is given by

$$\begin{aligned} S^{\mu\nu} &= \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 + i\epsilon)((k-p)^2 + i\epsilon)} = \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{k^\mu k^\nu}{[k^2 - 2xp\cdot k + xp^2 + i\epsilon]^2} \\ &= \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{k^\mu k^\nu}{[(k-xp)^2 + x(1-x)p^2 + i\epsilon]^2}. \end{aligned} \tag{A.9}$$

Shifting $k \to k + xp$, we can identify $\Delta = -x(1-x)p^2 - i\epsilon$ and find

$$\begin{aligned} S^{\mu\nu} &= \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{k^\mu k^\nu + x^2 p^\mu p^\nu + x(p^\mu k^\nu + k^\mu p^\nu)}{(k^2 - \Delta)^2} \\ &= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\frac{g^{\mu\nu}}{d} k^2 + x^2 p^\mu p^\nu}{(k^2 - \Delta)^2}, \end{aligned} \tag{A.10}$$

where we used that the linear terms in k vanish upon integration. For the quadratic terms we used Lorentz invariance, i.e.,

$$\int \frac{d^dk}{(2\pi)^d} k^\mu k^\nu f(k^2) = g^{\mu\nu} \int \frac{d^dk}{(2\pi)^d} \tilde{f}(k^2). \tag{A.11}$$

Upon contraction with $g^{\mu\nu}$ this gives

$$\int \frac{d^dk}{(2\pi)^d} k^2 f(k^2) = d \int \frac{d^dk}{(2\pi)^d} \tilde{f}(k^2) \tag{A.12}$$

and thus $\tilde{f}(k^2) = f(k^2)k^2/d$ in the integral. In the form (A.10) the master formula (A.1) applies.

For more complicated integrals the generalized Feynman parameterization reads

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \frac{\Gamma(m)}{\Gamma(m_1)\cdots\Gamma(m_n)} \int_0^1 dx_1 \cdots \int_0^1 dx_n\, \delta\Big(1 - \sum_{i=1}^n x_i\Big) \frac{\prod_{i=1}^n x_i^{m_i - 1}}{\Big[\sum_{i=1}^n x_i A_i\Big]^m}, \tag{A.13}$$

where $m = \sum_{i=1}^n m_i$. To derive this relation, we proceed by induction in n. The case $n = 2$ follows by taking derivatives of

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dy\, \delta(1 - x - y) \frac{1}{(xA + yB)^2} \tag{A.14}$$

with respect to A and B, which gives

$$\frac{1}{A^{m_1}B^{m_2}} = \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\Gamma(m_2)} \int_0^1 dx \int_0^1 dy\, \delta(1-x-y) \frac{x^{m_1-1}y^{m_2-1}}{(xA+yB)^{m_1+m_2}}. \tag{A.15}$$

In the induction step, we assume (A.13) for $n-1$, and then use (A.15) to combine A_n with the rest:

$$\begin{aligned}
\frac{1}{A_1^{m_1}A_2^{m_2}\cdots A_n^{m_n}} &= \frac{\Gamma(m)}{\Gamma(m-m_n)\Gamma(m_n)} \int_0^1 dx \int_0^1 dy\, \delta(1-x-y) x^{m_n-1} y^{m-m_n-1} \\
&\times \frac{\Gamma(m-m_n)}{\Gamma(m_1)\cdots\Gamma(m_{n-1})} \int_0^1 dx_1 \cdots \int_0^1 dx_{n-1} \frac{\delta\Big(1-\sum_{i=1}^{n-1} x_i\Big) \prod_{i=1}^{n-1} x_i^{m_i-1}}{\Big[xA_n + y\Big(\sum_{i=1}^{n-1} x_i A_i\Big)\Big]^m} \\
&= \frac{\Gamma(m)}{\Gamma(m_1)\cdots\Gamma(m_n)} \int_0^1 dy\, y^{m-m_n-1} \int_0^1 dx_1 \cdots \int_0^1 dx_n\, \delta(1-y-x_n) \\
&\times \delta\Big(1-\sum_{i=1}^{n-1} x_i\Big) \frac{\prod_{i=1}^{n} x_i^{m_i-1}}{\Big[x_nA_n + y\Big(\sum_{i=1}^{n-1} x_i A_i\Big)\Big]^m}.
\end{aligned} \tag{A.16}$$

The induction step follows by rescaling $x_i \to x_i/y$ for $i = 1, \ldots, n-1$ and the remaining integral

$$\begin{aligned}
\int_0^1 dy\, y^{-1} \delta(1-y-x_n)\, \delta\Big(1-\frac{1}{y}\sum_{i=1}^{n-1} x_i\Big) &= \int_0^1 dy\, y^{-1} \delta(1-y-x_n)\, y\delta\Big(y-\sum_{i=1}^{n-1} x_i\Big) \\
&= \delta\Big(1-\sum_{i=1}^{n} x_i\Big).
\end{aligned} \tag{A.17}$$

The upper integration boundaries are not affected and can be set to ∞ in intermediate steps to simplify the argument.

B. Feynman rules for derivative couplings

Feynman rules for derivative couplings are formally derived by Fourier transform to momentum space, just as for the standard case without derivatives, where a derivative ∂_μ acting on an incoming particle with momentum p gives $-ip_\mu$ and the opposite sign for an outgoing particle. Moreover, to facilitate calculations it is often useful to symmetrize the resulting Feynman rule. Let us consider as an example the interaction term

$$\delta\mathcal{L} = \frac{g}{4!}\phi^2(x)\Box\phi^2(x) \tag{B.1}$$

and transform it to momentum space according to

$$\phi(x) = \int_k e^{-ik\cdot x}\tilde{\phi}(k) = \int \frac{d^dk}{(2\pi)^d}e^{-ik\cdot x}\tilde{\phi}(k), \tag{B.2}$$

where k is an incoming momentum. In Fourier space the interaction becomes

$$\begin{aligned}\int d^dx\,\delta\mathcal{L} &= \int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\int d^dx\,\frac{g}{4!}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\big[-(k_3+k_4)^2\big]\tilde{\phi}(k_3)\tilde{\phi}(k_4)e^{-i(k_1+k_2+k_3+k_4)\cdot x}\\ &= -\frac{g}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\tilde{\phi}(k_1)\tilde{\phi}(k_2)(k_3+k_4)^2\tilde{\phi}(k_3)\tilde{\phi}(k_4)(2\pi)^d\delta^{(4)}(k_1+k_2+k_3+k_4).\end{aligned} \tag{B.3}$$

The disadvantage of this form is that two of the fields are singled out, so that in the application one would need to remember on which fields the derivatives act. It is much easier to symmetrize the Feynman rule using momentum conservation, i.e.,

$$\begin{aligned}\int d^dx\,\delta\mathcal{L} = -\frac{g}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}&\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)(2\pi)^d\delta^{(4)}(k_1+k_2+k_3+k_4)\\ &\times\frac{1}{3}\big[(k_1+k_2)^2+(k_1+k_3)^2+(k_1+k_4)^2\big],\end{aligned} \tag{B.4}$$

which is now completely symmetric in the three distinct momentum pairs. Let us consider the resulting amplitude for the four-point function, i.e., the scattering process

$$\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4), \tag{B.5}$$

with $p_{1,2}$ incoming and $p_{3,4}$ outgoing. The 4! is canceled by the combinatorial factor from all possible Wick contractions, just as for ϕ^4 theory. Accordingly, the amplitude becomes

$$\mathcal{M} = -\frac{g}{3}\big[(p_1+p_2)^2+(p_1-p_3)^2+(p_1-p_4)^2\big] = -\frac{g}{3}\big[p_1^2+p_2^2+p_3^2+p_4^2\big], \tag{B.6}$$

which on the mass shell, $p_i^2 = m^2$, reduces to $\mathcal{M} = -4gm^2/3$.

C. CCWZ construction of phenomenological Lagrangians

The construction of effective Lagrangians becomes more complicated if the symmetries of the full theory are not fully realized by the ground state, i.e., if a symmetry becomes spontaneously broken. In this case, a method to construct the effective Lagrangian was developed by Callan, Coleman, Wess, and Zumino (CCWZ) in Refs. [31,32]. Here, we follow the presentation from Ref. [33].

Suppose, the full theory is invariant under the group G, while the ground state is only invariant under the subgroup H of G, giving rise to $n = n_G - n_H$ Goldstone bosons, where n_G and n_H denote the number of generators. The Goldstone bosons are described by fields ϕ_i, collected in a vector $\Phi = (\phi_1, \ldots, \phi_n)$. This defines a vector space

$$M_1 \equiv \left\{\Phi : M^4 \to \mathbb{R}^n | \phi_i : M^4 \to \mathbb{R}\right\}, \tag{C.1}$$

with M^4 Minkowski space. The main point in the CCWZ construction amounts to establishing a connection between the so-called quotient G/H and the Goldstone-boson fields, in such a way that the effective Lagrangian can then be parameterized by resorting to a set of variables parameterizing the elements of G/H. The application described in a main text concerns low-energy QCD with $G = SU(3)_L \times SU(3)_R$ and $H = SU(3)_V$ ("Chiral perturbation theory"), another generalized realizations of the Higgs sector with $G = SU(2)_L \times U(1)_Y$ and $H = U(1)_{\text{EM}}$ ("Higgs effective field theory").

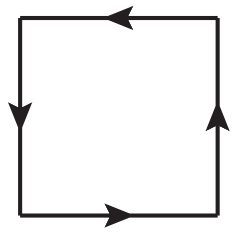
The aim is to find a mapping $\phi(g, \Phi)$ from $G \times M_1 \to M_1$ with the following properties

$$\begin{aligned} \phi(e, \Phi) &= \Phi \quad \forall\, \Phi \in M_1, \quad e \text{ identity of } G, \\ \phi(g_1, \phi(g_2, \Phi)) &= \phi(g_1 g_2, \Phi) \quad \forall\, g_1, g_2 \in G,\, \Phi \in M_1. \end{aligned} \tag{C.2}$$

Such a mapping is called an *operation* of G on M_1, and the second condition the *group-homomorphism property*. Note that we do not require this mapping to be linear, i.e., in general $\phi(g, \lambda\Phi) \neq \lambda\phi(g, \Phi)$, so the result will *not* define a representation.

Let us first consider $\Phi = 0$, for which all fields are mapped onto the origin in $\mathbb{R}^n$, which, in a theory with Goldstone boson only, can be interpreted as the ground-state configuration. Since the ground state is invariant under H, we require that $\phi(h, 0) = 0$ for $h \in H$. Next, we turn to the quotient G/H, which is defined as the set of all left cosets $\{gH | g \in G\}$. Here, the set $gH = \{gh | h \in H\}$ defines the left coset of $g \in G$, and the quotient is the set of all such cosets. An important property of this construction is that cosets either completely overlap or are completely disjoint.

Before proceeding, let us illustrate these properties using the symmetry group C_4 of a square with directed sides:



This group consists of four elements $C_4 = \{e, a, a^2, a^3\}$, where a can be interpreted as a rotation by $\pi/2$. The nontrivial subgroup is $H = \{e, a^2\}$, with left cosets

$$eH = \{e, a^2\} = a^2 H, \qquad aH = \{a, a^3\} = a^3 H. \tag{C.3}$$

The quotient G/H therefore consists of the two elements $\{e, a^2\}$ and $\{a, a^3\}$. Since the elements of G/H are completely disjoint, any element of a given coset uniquely represents the coset in which it appears. It is this property that is exploited in the CCWZ construction, to which we now return.

For $g \in G$ and $h \in H$ we have

$$\phi(gh, 0) = \phi(g, \phi(h, 0)) = \phi(g, 0), \tag{C.4}$$

i.e., the action on $\Phi = 0$ is identical among a given coset gH, which can be interpreted in such a way that the origin is mapped onto the same vector in $\mathbb{R}^n$. Second, the mapping is injective with respect to the elements of G/H (no two elements are mapped onto the same Φ), which can be seen as follows: consider $g, g' \in G$ with $g' \notin gH$ and let us assume $\phi(g, 0) = \phi(g', 0)$. Then

$$0 = \phi(e, 0) = \phi(g^{-1}g, 0) = \phi(g^{-1}, \phi(g, 0)) = \phi(g^{-1}, \phi(g', 0)) = \phi(g^{-1}g', 0), \tag{C.5}$$

which implies $g^{-1}g' \in H$, i.e., $g' \in gH$, in contradiction to the assumption, so that $\phi(g, 0) = \phi(g', 0)$ must be false. From this one concludes that there exists an *isomorphic mapping* between G/H and the Goldstone-boson fields. To account for the fact that they also depend on $x \in M^4$ (and are not just constant vectors in $\mathbb{R}^n$ as assumed so far), the cosets gH are also allowed to depend on x.

For the construction of the effective Lagrangian we need the transformation behavior of the Goldstone-boson fields under $g \in G$, which we can now study based on the isomorphism just established. To each Φ corresponds a coset $\tilde{g}H$ with some $\tilde{g}$. Let $f = \tilde{g}h \in \tilde{g}H$ denote a representative of this coset, i.e.,

$$\Phi = \phi(f, 0) = \phi(\tilde{g}h, 0). \tag{C.6}$$

Now,

$$\phi(g, \Phi) = \phi(g, \phi(\tilde{g}h, 0)) = \phi(g\tilde{g}h, 0) = \phi(f', 0) = \Phi', \qquad f' \in g(\tilde{g}H), \tag{C.7}$$

so, in order to obtain the transformed Φ' from a given Φ we simply need to multiply the left coset $\tilde{g}H$ representing Φ by g in order to obtain the new left coset representing Φ'. This procedure then determines the transformation behavior of the Goldstone bosons, leaving the task of finding a convenient parameterization of G/H.

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