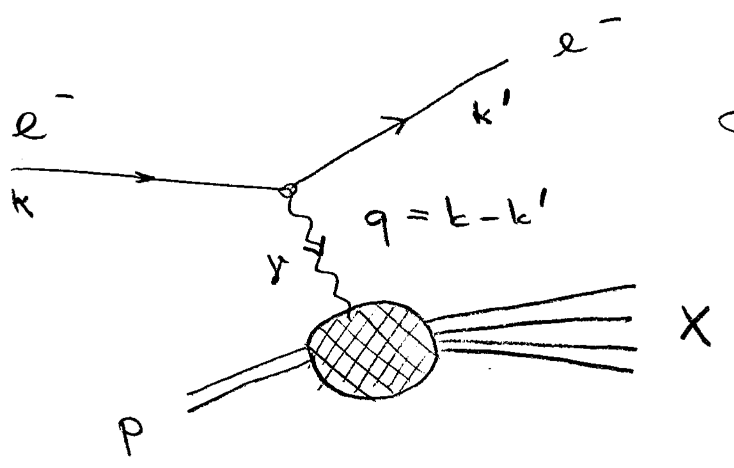
8. Deep inelastic scattering (DIS)

DIS refers to the process $e^- p \to e^- + X$ at high energies. At low energy, the scattering is elastic $e^- p \to e^- p$, but at high energies, one ends up with with a lot of hadrons.



$$Q^2 = -q^2$$

The amplitude is ($J^\mu_{e.m} = \sum_q e_q \bar{q} \gamma^\mu q$)

$$i\mathcal{M}(ep \to eX) = (-ie)\, \bar{u}(k') \gamma^\mu u(k) \frac{-i}{q^2}\, ie \int d^4x \sum_X e^{iq\cdot x} \langle X | J^\mu_{e.m}(x) | p \rangle$$

As in the case of $e^+e^- \to X$, let's use the optical theorem to rewrite the cross section.

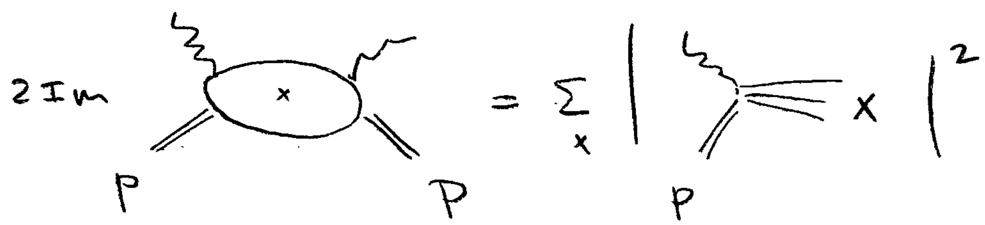
Consider

$$W^{\mu\nu}(q,P) = i\int d^4x\, e^{iqx} \langle P | T\{ J^\mu_{em}(x), J^\nu_{em}(x)\} | P\rangle$$

averaged over proton spins. From this matrix element, we obtain the Compton scattering amplitude:

$$i\mathcal{M}(\gamma p \to \gamma p) = (ie)^2\, \varepsilon^*_\mu\, \varepsilon_\nu\, (-iW^{\mu\nu}(q,P))$$

$$2\,\mathrm{Im}\,\mathcal{M}(\gamma p \to \gamma p) = \sum_X |\mathcal{M}(\gamma p \to X)|^2$$



or

$$2\,\mathrm{Im}\, W^{\mu\nu} = \sum_X \langle P | \tilde{J}^\mu(-q) | X\rangle\langle X | \tilde{J}^\nu(q) | P\rangle$$

Now consider the DIS cross section

$$\sigma = \frac{1}{2s} \int \frac{d^3k'}{2E_k (2\pi)^3} \sum_X \left| \mathcal{M}\left(e(k)\, p(P) \to e^-(k')\, X(p_X) \right) \right|^2$$

Neglect m_e, m_p

$$= \frac{1}{2s} \int \frac{d^3k'}{2E_k (2\pi)^3} \, e^4 \frac{1}{(Q^2)^2} \, \frac{1}{2} \sum_{\text{spins}} \bar{u}(k) \gamma_\mu u(k') \bar{u}(k') \gamma_\nu u(k)$$

$$\cdot \; 2 \, \mathrm{Im} \, W^{\mu\nu}(P, q).$$

$$\frac{1}{2} \sum_{\text{spins}} \bar{u}(k) \gamma_\mu u(k') \bar{u}(k') \gamma_\nu u(k) = \frac{1}{2} \mathrm{tr} \left[\not{k} \gamma_\mu \not{k}' \gamma_\nu \right]$$

$$= 2 \left(k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu} \, k \cdot k' \right)$$

Since $W^{\mu\nu}$ is the Compton amplitude, we must have $q^\mu W^{\mu\nu} = 0 = q^\nu W^{\mu\nu}$. This Ward identity simply states current conservation $\partial_\mu J^\mu = 0$.

The most general form of $W^{\mu\nu}$ consistent with this requirement is

$$W^{\mu\nu}(P,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(P,q)$$

$$+ \left(P_\mu - q^\mu \frac{P.q}{q^2}\right)\left(P_\nu - q_\nu \frac{P.q}{q^2}\right) W_2(P,q).$$

The following to variables (due to Bjorken) are often used

$s =$

$$x = \frac{Q^2}{2P.q} = \frac{2E_k E_k'(1-\cos\theta)}{2m_p(E_k - E_{k'})}$$

$$y = \frac{2P.q}{2P.k} = \frac{E_k - E_{k'}}{E_k} \cong \frac{Q^2}{x}\,\frac{1}{s} \quad ; \quad s = (P+k)^2 = 2m_p E_k$$

Jacobian:

$$\begin{vmatrix} \partial x/\partial E_{k'} & \partial x/\partial\cos\theta \\ \partial y/\partial E_{k'} & \partial y/\partial\cos\theta \end{vmatrix} = \begin{vmatrix} \cdots & \frac{-2E_k E_{k'}}{2m_p(E_k - E_{k'})} \\ -1/E_k & 0 \end{vmatrix}$$

$$= \frac{E_{k'}}{2m_p(E_k - E_{k'})} = \frac{2E_{k'}}{s\,y}$$

$$\int \frac{d^3k'}{2E_{k'}} = \int \frac{dE_{k'}}{2} E_{k'} \int_{-1}^{1} d\cos\theta \, 2\pi = \int dx \int dy \, \frac{y s \, 2\pi}{4}$$

The cross section is thus

$$\frac{d^2\sigma}{dx\,dy} = \frac{1}{\cancel{2}s} \frac{e^4}{Q^4} \frac{1}{(2\pi)^3} \cancel{2} (k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu} k k') \frac{2\pi}{4} y \cancel{s}$$
$$\cdot 2 \, \mathrm{Im} \, W^{\mu\nu}$$

$$= \frac{2\alpha^2 y}{(Q^2)^2} (k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu} k \cdot k') \, \mathrm{Im} \, W^{\mu\nu}$$

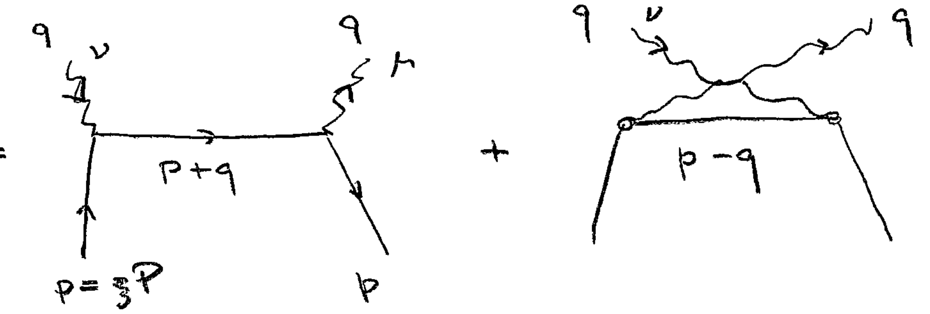
Also the lepton tensor vanishes when contracted with q^μ, so

$$\frac{d^2\sigma}{dx\,dy} = \frac{2\alpha^2 y}{(Q^2)^2} \Big[2 k \cdot k' \, \mathrm{Im} \, W_1 +$$
$$2 P \cdot k \, P k' \, \mathrm{Im} \, W_2 \Big]$$

$$= \frac{2\alpha^2 y}{(Q^2)^2} \left[Q^2 \, \mathrm{Im} \, W_1 + \frac{s^2}{2} (1-y) \, \mathrm{Im} \, W_2 \right]$$

$$[Q^2 = xys]$$

Before analyzing the hadronic matrix element, let's calculate $\hat{W}_{\mu\nu}$, the $W_{\mu\nu}$ for a quark instead of a proton state. Let's further write the quark momentum as $p = \xi P$. (we will later assume that the quark carries a fraction ξ of the proton momentum.)

$\hat{W}_{\mu\nu} =$ 

$$= i e_q^2 \, \frac{1}{2} \sum_s \bar{u}(p,s) \gamma^\mu \frac{i(\not{p}+\not{q})}{(p+q)^2 + i\varepsilon} \gamma^\nu u(p,s) + \begin{pmatrix} \mu \leftrightarrow \nu \\ q \leftrightarrow -q \end{pmatrix}$$

$$= - e_q^2 \, \frac{1}{2} \, \mathrm{tr}\left[\not{p} \gamma^\mu (\not{p} + \not{q}) \gamma^\nu \right] \frac{1}{2p\cdot q - Q^2 + i\varepsilon} + (\ldots)$$

Since $Q^2 > 0$, only the first diagram has an imaginary part.

$$\mathrm{Im}\left[\frac{1}{2p\cdot q - Q^2 + i\varepsilon} \right] = -\pi \, \delta(2\xi P\cdot q - Q^2) = -\frac{\pi}{ys} \delta(\xi - x)$$

$$\mathrm{Im}\, W^{\mu\nu} = \frac{\pi}{ys} e_q^2 \, \delta(\xi - x) \Big[- g_{\mu\nu} \, ys\xi + 4\xi^2 P_\mu P_\nu + \ldots \Big]$$

$$\mathrm{Im}\,\hat{W}_1 = \pi e_q^2\, \delta(\xi - x)\, \xi$$

$$\mathrm{Im}\,\hat{W}_2 = \frac{4\pi e_q^2}{ys}\, \xi^2\, \delta(\xi - x)$$

With this result at hand, we can now discuss the parton model. This model assumes that the proton is composed of partons (i.e. quarks and gluons). The partons carry fractions of the proton momentum. The proton scattering cross section is obtained by multiplying the probability $f_{q/p}(\xi)$ to find a quark with fraction ξ by the cross section for the quark to scatter.

Corrects the flux factor $\frac{1}{2s} \to \frac{1}{2\xi s}$

So : $$\mathrm{Im}\,W_1 = \sum_q \int_0^1 \frac{d\xi}{\xi}\, f_{q/p}(\xi) \cdot \mathrm{Im}\,\hat{W}_1(\xi P, Q^2)$$

$$= \pi \sum_q e_q^2\, f_{q/p}(x)$$

$$\mathrm{Im}\,W_2 = \frac{4\pi e_q^2}{ys}\, x \sum_q e_q^2\, f_{q/p}(x)$$

The relation

$$\mathrm{Im}\, W_1 = \frac{ys}{4x}\, \mathrm{Im}\, W_2$$

is called the Callan - Gross relation.

It is characteristic for spin - $\frac{1}{2}$ partons.
For spin 1, one obtains $W_2 = 0$. The fact that the relation was supported experimentelly was taken as an indication that the quarks could be the partons (partons were introduced before QCD.)

Plugging into the formula for the cross section, one has

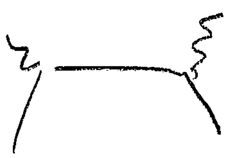
$$\frac{d\sigma}{dx\,dy} = \frac{2\pi\alpha^2 s}{Q^4}\left[1+(1-y)^2\right]\sum_q e_q^2\, x\, f_{q/p}(x)$$

While the x-dependence of the cross section is given by the a-priori unknown function $f_{q/p}(x)$, the Q^2-dependence is predicted.

The data indeed were compatible with the $1/Q^4$ scaling of the cross section supporting the parton model interpretation.

The $1/Q^4$ scaling is called Bjorken scaling. Higher orders in QCD lead to logarithmic corrections to the scaling law.

Since the hadronic matrix element is given by a product of currents, it is tempting to try to analyze it using the OPE. However the fact that the proton has a large energy ruins the expansion



$$\sim \frac{1}{2P\cdot q - Q^2} \overset{?}{=} \frac{1}{-Q^2}\left\{1 + \frac{2Pq}{Q^2} + \dots\right\}$$

Since P has large energy, we cannot neglect $2P\cdot q \sim Q^2$ and the expansion breaks down.

The traditional way of dealing with this problem is to expand anyway and then to resum all those higher-dim operators which are not suppressed. This is quite cumbersome.

Our effective theory was designed to analyze processes with energetic particles and will allow us to derive the result in a quite straightforward way.