8.2. Parton Distribution Functions (PDFs)

To calculate DIS or any hadron-collider cross-section we need the non-perturbative PDFs as input.

We now derive some properties of these functions and then discuss their determination

The PDFs fulfill the following sum rules

Momentum sum rule:

$$\sum_i \int_0^1 d\xi \, \xi \, f_i(\xi) = 1$$

Flavor conservation. For a proton

$$\int_0^1 d\xi \sum_i \left(f_u(\xi) - f_{\bar{u}}(\xi) \right) = 2$$

$$\int_0^1 d\xi \sum_i \left(f_d(\xi) - f_{\bar{d}}(\xi) \right) = 1$$

$$\int_0^1 d\xi \sum_i \left(f_s(\xi) - f_{\bar{s}}(\xi) \right) = 0$$

In the parton-model interpretation, these are simple to understand. The momentum sum rule simply states, that the total momentum of the proton is given by the sum of all parton momenta.

The flavor sum rules state that the proton contains two u-quarks and $\bar{u}u$ pairs, etc.

Let us derive the flavor sum rule from the operator definition:

$$f_q(\xi) = \int \frac{ds}{2\pi}\, e^{-i\bar{n}\cdot P \xi s} \langle p | \bar{\chi}_q(s\bar{n}) \frac{\not{n}}{2} \chi(0) | p \rangle$$

$$= \int \frac{ds}{2\pi}\, e^{-i\bar{n}\cdot P \xi s} \langle p | \bar{q}(s\bar{n}) \frac{\not{n}}{2} [s\bar{n}, 0]\, q(0) | p \rangle$$

We have

$$f_{\bar{q}}(\xi) = -\, f_q(-\xi)$$

So

$$\int_0^1 d\xi \, [f_q(\xi) - f_{\bar{q}}(\xi)]$$

$f_q(\xi) = 0$ for $|\xi| > 1$

$$= \int_{-1}^{1} d\xi \, f_q(\xi) = \int_{-\infty}^{\infty} d\xi \, f_q(\xi)$$

$$= \int d\xi \int \frac{ds}{2\pi} e^{i\bar{n}P\xi s} \langle p | \bar{q}(s\bar{n}) \frac{\bar{\not{n}}}{2} [s\bar{n}, 0] q(0) | p \rangle$$

$$= \frac{1}{\bar{n}\cdot P} \langle p | \bar{q}(0) \frac{\bar{\not{n}}}{2} q(0) | p \rangle$$

$$\langle p | \bar{q}\gamma^0 q | p \rangle = (\#q\text{'s} - \#\bar{q}\text{'s}) \cdot 2E$$

$$\langle p | \bar{q}\gamma^\mu q | p \rangle = (\#q\text{'s} - \#\bar{q}\text{'s}) \, 2P^\mu$$

$$\Rightarrow \int_0^1 d\xi \, [f_u(\xi) - f_{\bar{u}}(\xi)] = 2$$

So we have derived the flavor sum rules. Note that an integral over the PDF lead to a <u>local</u> operator.

More generally, let us consider the moments

$$M_q^N = \int_0^1 \frac{d\xi}{\xi}\, \xi^N \left[f_q(\xi) + (-1)^N f_{\bar q}(\xi) \right]$$

$$= \int_{-\infty}^{\infty} \frac{d\xi}{\xi}\, \xi^N f_q(x)$$

$$= \int_{-\infty}^{\infty} d\xi\, \xi^{N-1} \int \frac{ds}{2\pi}\, e^{-i\bar n\cdot p \xi s} \langle p | \bar\chi(s\bar n) \frac{\not{\bar n}}{2} \chi(0) | p \rangle$$

$$= \int_{-\infty}^{\infty} d\xi \int \frac{ds}{(2\pi)} \left[\left(\frac{i\partial_s}{\bar n\cdot p} \right)^{N-1} e^{-i\bar n\cdot p \xi s} \right] \langle p | \bar\chi(s\bar n) \frac{\not{\bar n}}{2} \chi(0) | p \rangle$$

$$= \int_{-\infty}^{\infty} d\xi \int \frac{ds}{2\pi}\, e^{i\bar n\cdot p \xi s} \left(\frac{-i\partial_s}{\bar n\cdot p} \right)^{N-1} \langle p | \bar\chi(s\bar n) \frac{\not{\bar n}}{2} \chi(0) | p \rangle$$

$$(-i\partial_s)\, \bar q(s\bar n)[s\bar n, 0] q(0) \Big|_{s=0}$$

$$= \{ -i\partial_s \bar q(s\bar n) \} [s\bar n, 0] q(0) + \bar q(s\bar n)\, g A(s\bar n) [s\bar n, 0] q(0)$$

$$= \bar q(s\bar n)(-i\bar n\cdot \overleftarrow{D})[s\bar n, 0] q(0)$$

$$= \left(\frac{1}{\bar n\cdot p} \right)^{N-1} \langle p | \bar q(0) (i\bar n\cdot D)^{N-1} \frac{\not{\bar n}}{2} q(0) | p \rangle$$

So moments of the PDFs correspond to local operators which can be calculated using lattice gauge theory. Unfortunately only $N = 2, 3$ is feasible. The reason is that these operators need to be renormalized. On the lattice with spacing a

$$\langle p | \bar{q} (iD)^{N-1} \frac{\not{n}}{2} q | p \rangle \sim \left(\frac{1}{a} \right)^{N-2}$$

the strong divergence makes it hard to extract the finite piece numerically.

Not only the local operators, also the non-local PDF operators need to be renormalized.

For the PDFs

$$\begin{pmatrix} f_q^{ren}(\xi, \mu) \\ f_g^{ren}(\xi, \mu) \end{pmatrix} = \int_\xi^1 \frac{dz}{z} \begin{pmatrix} Z_{qq}(z) & Z_{qg}(z) \\ Z_{gq}(z) & Z_{gg}(z) \end{pmatrix} \begin{pmatrix} f_q(\frac{\xi}{z}) \\ f_g(\frac{\xi}{z}) \end{pmatrix}$$

This is the usual $O^{ren} = Z \; O^{bare}$ relation.

Since we have two operators, they <u>mix</u> under renormalization and Z is a matrix. Furthermore, also the operators of different ξ mix, which gives the convolution.

The renormalized PDFs depend on the renormalization scale. From the fact, that the bare PDFs are scale independent, one obtains the renormalization group equation

$$\frac{d}{d\ln\mu}\begin{pmatrix} f_q^{ren}(\xi,\mu) \\ f_g^{ren}(\xi,\mu)\end{pmatrix} = \int_0^1 \frac{dz}{z} \overbrace{\begin{pmatrix} P_{qq}(z) & P_{qg}(z) \\ P_{gq}(z) & P_{gg}(z)\end{pmatrix}}^{\Gamma} \begin{pmatrix} f_q^{ren}(\frac{\xi}{z},\mu) \\ f_g(\frac{\xi}{z},\mu)\end{pmatrix}$$

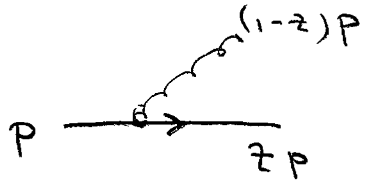
$\Gamma = - Z \otimes \frac{d}{d\ln\mu} Z^{-1}$ is the anomalous dimension which governs the scale dependence.

While we cannot calculate the PDFs in perturbation theory, we can calculate the scale dependence.

The above equation is called the DGLAP equation. (Altarelli, Parisi; Gribov, Lipatov; Dokshitzer '77)

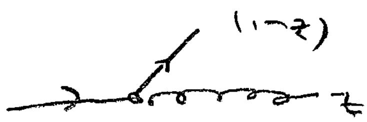
The kernels are $P_{ij}(z) = \frac{\alpha}{2\pi} P_{ij}^{(0)}(z) + \ldots$

$$P_{qq}^{(0)} = C_F \left[\left(\frac{1+z^2}{1-z} \right)_+ + \frac{3}{2} \delta(1-z) \right]$$

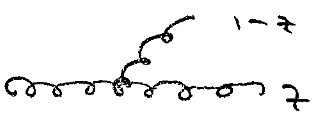


$$P_{qg}^{(0)} = T_F \left(z^2 + (1-z) \right)$$

$$P_{gq}^{(0)} = C_F \, \frac{1 + (1-z)^2}{z}$$



$$P_{gg}^{(0)} = C_A \left[z \left(\frac{1}{1-z} \right)_+ + \frac{1-z}{z} + z(1-z) + \beta_0 \delta(1-z) \right]$$



They are also called splitting functions and are known to NNLO (Moch, Vermaseren and Vogt '04)

To solve the evolution equation, people usually go to moment space

$$M_i^N(\mu) = \int d\xi\, \xi^{N-1} f_i(\xi,\mu)$$

Then the problem reduces to the solution of the RG equation for the local operators.

At the end, one has to transform back:

$$f_i(\xi,\mu) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dN\, x^{-N} M_i^N(\xi,\mu)$$

Numerically, this is tricky, but there exists commercial grade numerical code to do it.

Let us now discuss the determination of the PDFs. We had obtained

$$\frac{d^2\sigma}{dx\,dy} = \frac{2\pi\alpha^2 s}{Q^4}\left[1+(1-y)^2\right]\sum_q e_q^2\, x\, f_q(x,\mu)$$

Define $F_2(x) = \sum_q e_q^2 f_q(x,\mu)$.

The natural choice for μ is $\mu = Q$.

Measuring $F_2(x)$ determines one linear combination of PDFs.

$$F_2 = x\left[\frac{4}{9} f_u(x) + \frac{1}{9} f_d(x)\right] + (\text{"sea-quarks"})$$

To get more information, scatter on neutron and use isospin

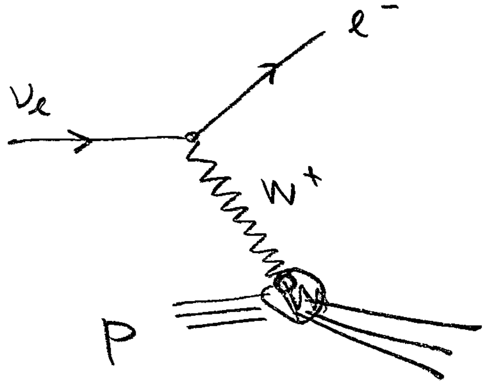
$$F_2^n = x\left[\frac{4}{9} f_u^n(x) + \frac{1}{9} f_d^n(x)\right] + (\text{"sea"})$$

$$= x\left[\frac{4}{9} f_d(x) + \frac{1}{9} f_u(x)\right] + (\text{"sea"})$$

(Experimentally one uses deutrons

$$F_2^d \cong \frac{1}{2}\left(F_2^p(x) + F_2^n(x)\right)$$

We also need the PDFs of s-quarks, and u-, d-, s-anti-quarks. To get these, we need something which interacts differently with quarks and anti-quarks. Use W's:



Finally, we need the gluon distribution. It can be inferred by measuring at different Q^2 and relating the PDFs by evolution.

In practice, people start with a parametrization of the PDFs at some scale Q_0, e.g.

$$f_q(\xi, Q_0) = A_q x^{a_q}(1-x)^{b_q}\left[1 + c_q\sqrt{x} + \dots\right]$$

Then they evolve it to all Q^2 where exp. data is available and do a global fit to determine all parameters. There are a number of groups (MSTW, CTEQ, NNPDF, Alekhin, ...) doing such fits, and modern results also include a way to estimate the uncertainty. See figures for some results.