8.1. Factorization of the DIS cross section

Let us now analyze the process in SCET and derive a factorization theorem for the cross section. It is most convenient to work in the Breit frame, where

$$q^\mu = \frac{Q}{2}\left(\bar{n}^\mu - n^\mu\right) = Q\,(0,0,0,-1)$$

$$q^2 = -Q^2 \quad \checkmark$$

$$P^\mu = \bar{n}\cdot P\,\frac{n^\mu}{2} + \frac{M_P^2}{\bar{n}\cdot P}\,\frac{\bar{n}^\mu}{2}$$

$$P^2 = M_P^2\,\frac{2\bar{n}\cdot n}{4} = M_P^2 \quad \checkmark$$

$$x = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{\bar{n}\cdot P} + O\left(m_P^2/Q^2\right)$$

Recall that

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1 + \left(P^\mu - \frac{q^\mu q\cdot P}{q^2}\right)\left(P^\nu - \frac{q^\nu q\cdot P}{q^2}\right) W_2$$

$$= \left[-g^{\mu\nu} + \frac{(\bar{n}^\mu - n^\mu)(\bar{n}^\nu - n^\nu)}{4}\right] W_1 + \frac{(\bar{n}\cdot P)^2}{16}\,(n^\mu + \bar{n}^\mu)(n^\nu + \bar{n}^\nu)\, W_2$$

The tensor $W^{\mu\nu}$ is defined as

$$W^{\mu\nu} = \frac{1}{2}\sum_s \langle P, s \,|\, T^{\mu\nu} \,|\, P, s \rangle$$

$$T^{\mu\nu} = i \int d^4x \; e^{iqx} \; T[\, J^{\dagger}_{\mu}(x) \, J_{\nu}(0) \,]$$

One way to analyze $T^{\mu\nu}$ would be to expand it in a series of local operators, however because we later take the proton matrix element operators such as

$$\frac{1}{Q^n}\langle \bar{q} (\bar{n}\cdot D)^n q \rangle \sim \left(\frac{\bar{n}\cdot p}{Q}\right)^n = x^n$$

are not suppressed. This situation, where we have very energetic particles, is precisely what SCET was designed for. Because the $\bar{n}\cdot\partial$ derivatives on collinear fields are not suppressed the operators are non-local along the corresponding light-cone direction. To analyze $T^{\mu\nu}$ in this effective theory, we now write down the most general

Leading power operator in the EFT.

First, since we average over spins, we can write

$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{(\bar{n}^\mu - n^\mu)(\bar{n}^\nu - n^\nu)}{4}\right) T_1$$

$$\frac{(\bar{n}\cdot P)^2}{16} (n^\mu + \bar{n}^\mu)(n^\nu + \bar{n}^\nu)\, T_2$$

where T_1 and T_2 are scaler operators in SCET.

It is a bit strange to include a factor $\bar{n}\cdot P$ in the operator $T^{\mu\nu}$, since it does not know about the proton. We do it so that

$$\langle p | T_1 | p \rangle = W_1 \; ; \; \langle p | T_2 | p \rangle = W_2 .$$

So now we should write down operators. As building blocks, we use

$$\chi(x) = W_c^\dagger \xi_c(x)$$

$$\mathcal{B}_\perp^\mu(x) = W_c^\dagger(x)\, \bar{n}_\nu \left[G_c^{\nu\mu}(x) - G_c^{\nu\alpha}(x)\, n^\alpha \frac{\bar{n}^\mu}{2} \right] W_c(x)$$

Remember that $A_c^\mu \sim (n\cdot A, \bar{n} A, A_\perp^\mu)$
$$\sim (\lambda^2, 1, \lambda)$$

Thus $\bar{n}_\nu G_\perp^{\nu\mu} \sim \lambda$

$$\bar{n}_\nu n_\mu G^{\mu\nu} \sim \lambda^2$$

$$G_\perp^{\mu\nu} \sim \lambda^2$$

So at leading power only $\mathcal{B}_\perp^\mu$ can enter.

The lowest order operator is $\mathbb{1}$, however it does not have an imaginary part. There are no scalar operators with one field. The first nontrivial possibilities are

$$O_1 = \bar{\chi}(s\bar{n})\frac{\bar{\not{n}}}{2}\chi(t\bar{n}) \sim O(\lambda^2)$$

$$O_2 = \mathrm{tr}\left[\mathcal{B}_\perp^\mu(s\bar{n})\,\mathcal{B}_\perp^\mu(t\bar{n})\right] \sim O(\lambda^2)$$

Note that $\not{n}\chi = 0$, $\frac{\not{n}\bar{\not{n}}}{4}\chi = \chi$:

$$\bar{\chi}\chi = \bar{\chi}\frac{\not{n}\bar{\not{n}}}{4}\chi = 0$$

$$\bar{\chi}\not{n}\chi = 0$$

$\bar{\chi}\gamma_5\frac{\bar{\not{n}}}{2}\chi$ violates parity.

So O_1 and O_2 are the only operators at leading power. ~~Because~~ of translation invariance only the distance between the fields matters, so we can set $t=0$ without loss of generality. So, up to power suppressed terms

$$T_1 = \int ds \Big\{ C_{1q}(s,Q^2)\, \bar{\chi}(s\bar{n}^\mu) \frac{\bar{n\!\!\!/}}{2} \chi(0)$$
$$+ C_{1g}(s,Q^2)\, \mathrm{tr}\left[B_\perp^\alpha(s\bar{n}^\mu) B_{\perp\alpha}(0) \right] \Big\}$$

and analogously for T_2, with different coefficients $C(s,Q^2)$

Now we decouple soft gluons $\quad x^- = \bar{n}\cdot x \frac{n^\mu}{2}$

$$\chi(x) \longrightarrow S_n(x_-)\, \chi^{(0)}(x)$$

it follows $\quad \chi(s\bar{n}) \longrightarrow S_n(0)\, \chi^{(0)}(x)$

$$B_\perp^\mu(x) \longrightarrow S(x_-)\, B_\perp^\mu(x)\, S^+(x_-)$$

So $\bar{\chi}(s\bar{n})\frac{\bar{n}\!\!\!/}{2}\chi(0)$

$$\longrightarrow \bar{\chi}^{(0)}(s\bar{n})\frac{\bar{n}\!\!\!/}{2}\underbrace{S_n^{+}(0)S_n(0)}_{=1}\chi(0)$$

Similarly, the soft gluons decouple from $\mathrm{tr}[B_\perp^{\alpha}B_\alpha]$.

So we have factorized the cross section!

Now we just need to understand what the factorization theorem means .... To get an understanding, let's calculate $\hat{W}_{q1,2}$, the quark matrix element of $T_{1,2}$:

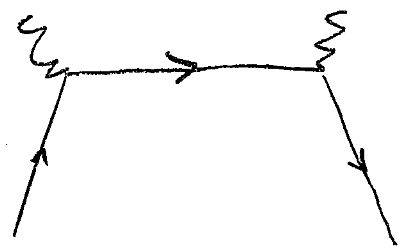
$$\frac{1}{2}\sum_{\text{spin}}\langle q(p)|\,T_1\,|q(p)\rangle =$$

$$\int ds\, C_{1q}(s,Q^2)\, e^{is\bar{n}\cdot p}\ \frac{1}{2}\sum_{\text{spin}}\bar{u}(p)\frac{\bar{n}\!\!\!/}{2}u(p)$$

$$= \tilde{C}_{1q}(\bar{n}\cdot p,Q^2)\ \frac{1}{2}\mathrm{tr}\Big[\overset{\approx\, \bar{n}\cdot p\,\frac{n\!\!\!/}{2}}{\overset{\shortparallel}{p\!\!\!/}}\ \frac{\bar{n}\!\!\!/}{2}\Big] = \tilde{C}_{1q}(\bar{n}\cdot p,Q^2)\ \bar{n}\cdot p$$

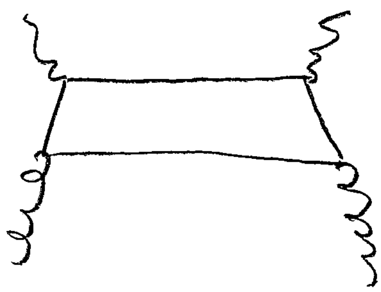
So $\tilde{C}_{1q}(\bar{n}\cdot p, Q^2)\,\bar{n}\cdot p = \hat{W}_{1q}(\bar{n}\cdot p, Q^2)$.

The Fourier transformed Wilson coefficient is just the partonic process



$\Rightarrow \tilde{C}_{1q}, \tilde{C}_{2g}$

The gluonic coefficient is obtained from



+ others $\Rightarrow C_{1g}, C_{2g}$ are $O(\alpha_s)$

Next, let's consider the proton matrix elements.

One defines

$$\frac{1}{2}\sum_{\text{spin}} \langle P \,|\, \bar{\chi}(s\bar{n})\frac{\bar{\not{n}}}{2}\chi(0)\,|\, P\rangle$$

$$= \bar{n}\cdot P \int d\xi\, f_q(\xi)\, e^{i\xi\bar{n}\cdot P s}.$$

and

$$\frac{1}{2}\sum_{\text{spin}} \langle P | B_\perp^{a}(s\bar{n}) B_{\perp\alpha}(0) | P \rangle$$

$$= \int d\xi \; \xi (\bar{n}\cdot P)^2 f_g(\xi)\, e^{i\xi \bar{n}\cdot P s}$$

The integrations run from $-1 < \xi < 1$. The function $f_q(\xi)$ picks up a contribution when the field χ carries away a fraction ξ of the proton's momentum. Negative values correspond to the anti-particle case

$$\bar{f}_q(\xi) = f_{\bar{q}}(\xi) = -f_q(-\xi)$$

$$f_g(\xi) = f_g(-\xi) = \bar{f}_g(\xi)$$

Now plug the expressions for the matrix elements into the expressions for W_1 and W_2:

$$W_1 = \int ds\, C_{1q}(s, Q^2) \int d\xi\, \bar{n}\cdot p\, e^{i\xi\bar{n}\cdot p} f_q(\xi) + \text{"glue"}$$

$$= \int_{-1}^{1} d\xi\, \tilde{C}_{1q}(\xi\bar{n}\cdot p, Q^2)\, \bar{n}\cdot p\, f_q(\xi) + \text{"glue"}$$

$$= \int_{-1}^{1} \frac{d\xi}{\xi} \hat{W}_{1q}(\xi\bar{n}\cdot p, Q^2)\, f_q(\xi) + \text{"glue"}$$

$$\boxed{\begin{aligned} W_i &= \sum_q \int_0^1 \frac{d\xi}{\xi} \hat{W}_{iq}(\xi\bar{n}\cdot p, Q^2)\left[f_q(\xi) + f_{\bar{q}}(\xi)\right] \\ &+ \int_0^1 \frac{d\xi}{\xi} \hat{W}_{ig}(\xi\bar{n}\cdot p, Q^2)\, f_g(\xi) \quad ; \; i = 1, 2 \end{aligned}}$$

$$\frac{d^2\sigma}{dx\,dy} = \frac{2\alpha^2 y}{(Q^2)^2}\left[Q^2\, \mathrm{Im}\, W_1 + \frac{s^2}{2}(1-y)\, \mathrm{Im}\, W_2\right]$$

The hadronic part is real, so that the imaginary part of W_1 and W_2 is obtained by taking the imaginary part of the partonic $\hat{W}_{iq}$ and $\hat{W}_{ig}$.

We are summing over different quark flavors $q = u, d, s, \ldots$

To summarize: we have factored the cross section into a set of non-perturbative parton distribution functions (PDFs) $f_i(\xi)$ which are convoluted with partonic cross section. To lowest order in α_s, we reproduce the parton model result, but our formula gives operator definitions for the PDFs and allows us to systematically include higher orders in α_s.