7.3. Generalization to QCD

The method for constructing the effective theory for QCD is analogous to the scalar case. In particular, the same momentum regions appear, since only the numerators of the diagrams differ between ϕ^3 and QCD.

The two complications with respect to the scalar case are that not all components of the quark and gluon fields have the same scaling and that we need to ensure gauge invariance, in particular also for the non-local operators.

To make things simpler, let's only consider one type of collinear field $p_c^\mu \sim (\lambda^2, 1, \lambda)$ for the moment. Let's look at the fermion field. Split

$$\psi_c(x) = \xi(x) + \eta(x) \;; \quad \xi = P_+ \psi_c = \frac{\not{n}\not{\bar{n}}}{4}\psi_c$$
$$\eta = P_- \psi_c = \frac{\not{\bar{n}}\not{n}}{4}\psi_c$$

<u>Note</u>: $\not{n}\xi = 0 \;; \quad \not{\bar{n}}\eta = 0$

$$P_+^2 = \frac{\not{n}\not{\bar{n}}}{4}\frac{\not{n}\not{\bar{n}}}{4} = -\frac{\not{n}\not{n}}{4}\frac{\not{\bar{n}}\not{\bar{n}}}{4} + \frac{\not{n}\not{\bar{n}}}{4}\frac{2n\cdot\bar{n}}{4} = \frac{\not{n}\not{\bar{n}}}{4} = P_+$$

$$P_+ + P_- = \frac{\not{n}\not{\bar{n}}}{4} + \frac{\not{\bar{n}}\not{n}}{4} = \frac{2n\cdot\bar{n}}{4} = 1 .$$

$$\langle 0 | T\{ \xi(x) \bar{\xi}(0) \} | 0 \rangle = \frac{\not{n}\not{\bar{n}}}{4} \langle 0 | T\{ \psi(x) \bar{\psi}(0) \} | 0 \rangle \frac{\not{\bar{n}}\not{n}}{4}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 + i\varepsilon} e^{-ipx} \underbrace{\frac{\not{n}\not{\bar{n}}}{4} \not{p} \frac{\not{\bar{n}}\not{n}}{4}}_{= \frac{\not{n}}{2} \bar{n}\cdot p} \sim \lambda^2 \lambda^2 \frac{1}{\lambda^2}$$

$$\Rightarrow \xi(x) \sim \lambda .$$

The same argument gives

$$\eta(x) \sim \lambda^2 .$$

For the soft quark field

$$q_s \bar{q}_s \sim (\lambda^2)^4 \frac{1}{\lambda^4} \lambda^2 = \lambda^6$$

$$q_s \sim \lambda^3$$

For the gluon field

$$\langle 0 | T\{ A^\mu(x) A^\nu(0) \} | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{p^2} \left(-g^{\mu\nu} + \xi \frac{p^\mu p^\nu}{p^2} \right)$$

$$\Rightarrow A_c^\mu \sim p_c^\mu \quad ; \quad A_s^\mu \sim p_s^\mu$$

$$\rightarrow \bar{n}\cdot A_{c_1} \sim 1 \; ; \; n \cdot A_{c_1} \sim \lambda^2 \; ; \; A_\perp^\mu \sim \lambda$$

$$A_s^\mu \sim \lambda^2 .$$

The collinear fermion Lagrangian has a special form since the two components encoded in η are of a higher power than the ξ components and can be integrated out. With $\not{n}\xi = 0$, $\not{\bar{n}}\eta = 0$, we find

$$\mathcal{L}_c = \bar{\Psi}_c \, i\not{D}_c \, \Psi = (\bar{\xi} + \bar{\eta})\left[\frac{\not{n}}{2} i\bar{n}\cdot D + \frac{\not{\bar{n}}}{2} in\cdot D + i\not{D}_\perp\right](\xi+\eta)$$

$$= \bar{\xi}\frac{\not{\bar{n}}}{2} n\cdot D\,\xi + \bar{\xi}\, i\not{D}_\perp \eta + \bar{\eta}\, i\not{D}_\perp \xi + \bar{\eta}\frac{\not{n}}{2}\bar{n}\cdot D\,\eta$$

Since the action is quadratic, we can integrate out η exactly. A short-cut for obtaining the result is to plug the solution of the equation of motion back into $\mathcal{L}_c$. The EoM's are

$$\frac{\not{\bar{n}}}{2} n\cdot D\,\xi = -i\not{D}_\perp \eta$$

$$i\not{D}_\perp \xi = -\frac{\not{n}}{2}\bar{n}\cdot D\,\eta$$

$$\Rightarrow \quad \frac{\not{\bar{n}}}{2} i\not{D}_\perp \xi = -\frac{\not{\bar{n}}\not{n}}{2}\bar{n}\cdot D\eta = -\bar{n}\cdot D\,\eta$$

$$\Rightarrow \quad \eta = -\frac{\not{\bar{n}}}{2i\bar{n}\cdot D} i\not{D}_\perp \xi \; ; \; \bar{\eta} = -\bar{\xi}\, i\overleftarrow{\not{D}}_\perp \frac{\not{\bar{n}}}{2i\bar{n}\cdot\overleftarrow{D}}$$

Plug in:

$$\mathcal{L}_c = \bar\xi_c \frac{\not n}{2} n\cdot D\, \xi_c + \bar\xi\, i\not D_\perp \frac{1}{i\bar n\cdot D}\, i\not D_\perp \frac{\not{\bar n}}{2}\,\xi$$

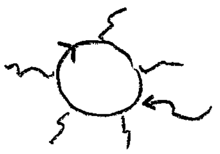
$$+ \bar\xi\, i\overleftarrow{\not D} \frac{1}{i\bar n \overleftarrow{D}}\, i\not D_\perp \frac{\not n}{2}\,\xi$$

$$+ \bar\xi\, i\overleftarrow{\not D} \frac{\not n}{2 i\bar n\overleftarrow{D}} \underbrace{\frac{\not n}{2}\, i\bar n D\, \frac{\not n}{2 i\bar n \overrightarrow{D}}}_{\frac{\not n \not{\bar n}}{4}\, \hat{=}\, 1} i\not D_\perp\, \xi$$

$$= \bar\xi_c \frac{\not n}{2} n\cdot D\, \xi_c + \bar\xi\, i\not D_\perp \frac{1}{i\bar n D \mp i\varepsilon}\, i\not D_\perp \frac{\not n}{2}\,\xi$$

sign will not matter. $\bar n\cdot D \sim E$ is large

If one integrate out the field η, one further obtains a determinant $\det(i\bar n\cdot D)$. This corresponds simply to a trivial overall factor in the path integral. To see this, note that it is gauge invariant, but independent of the gluon field in light-cone gauge $\bar n\cdot A = 0$. [Alternatively, it is easy to see that all loop diagrams



$\frac{1}{\bar n p + i\varepsilon}$

vanish, because all poles are on the same side of the $n\cdot p$ axis.]

While the collinear quark Lagrangian has this somewhat complicated form, the collinear gluon Lagrangian is just the QCD expression with $A_\mu \to A_\mu^c$.

The same is true for the soft Lagrangian

$$\mathcal{L}_s = \bar{q}_s i \not{D}_s q_s - \frac{1}{4}(F_{\mu\nu}^{sa})^2$$

where $iD_s = i\partial + A_s$

$$i g_s F_{\mu\nu}^a t^a = [iD_s^\mu, iD_s^\nu]$$

So our effective Lagrangian consists of several copies of QCD, for each collinear sector and for the soft momentum region.

What is still missing are the soft-collinear interactions. The general construction is somewhat complicated (see hep-ph/0211358 by Beneke and Feldmann) but we only need the leading power Lagrangian to derive factorization in the limit $E \to \infty$.

To get the leading interactions, remember that $(n\cdot A_{c_1}, \bar{n} A_{c_1}, A_{c\perp}) \sim (\lambda^2, 1, \lambda)$

$$(n A_s, n A_s, A_{s\perp}) \sim (\lambda^2, \lambda^2, \lambda^2)$$

$$\xi \sim \lambda \qquad q_s \sim \lambda^3$$

The leading soft-collinear interactions can be obtained by replacing $\phi_{c_1}(x) \to \phi_s(x_-)$ in the collinear Lagrangian

E.g. $-\frac{g}{3!}\int d^4x\, \phi_{c_1}^3(x) \longrightarrow -\frac{g}{2!}\int d^4x\, \phi_{c_1}^2(x)\,\phi_s(x_-)$

- Since q_s is of a higher power than ξ, interactions with soft quarks do not appear at leading order.
- Only the $n\cdot A_s$ component of the soft gluon field is not power suppressed, so only this component enters the leading soft-collinear interactions. So we can replace

$$A_c^\mu(x) \to (n\cdot A_c(x) + n\cdot A_s(x_-))\frac{\bar{n}^\mu}{2} + \bar{n}\cdot A_c(x)\frac{n^\mu}{2} + A_{c\perp}^\mu(x)$$

in the collinear Lagrangian.

To summarize, our Lagrangian is

$$\mathcal{L}_{SCET} = \bar{q}\, i\not{D}_s\, q + \bar{\xi}\frac{\not{\bar{n}}}{2}\left[n\cdot D + i\not{D}_{c\perp}\frac{1}{i\bar{n}\cdot D_c}\, i\not{D}_{c\perp}\right]\xi$$

$$-\frac{1}{4}\left(F^{sa}_{\mu\nu}\right)^2 - \frac{1}{4}\left(F^{ca}_{\mu\nu}\right)^2$$

where
$$iD_s = i\partial_\mu + g A_{s\mu}$$
$$iD_c = i\partial_\mu + g A_{c\mu}$$
$$in\cdot D = in\cdot\partial + g\, n\cdot A_c(x) + g\, n\cdot A_s(x_-)$$

and
$$ig F^s_{\mu\nu} = [iD^s_\mu, iD^s_\nu]$$
$$ig F^c_{\mu\nu} = [iD_\mu, iD_\nu]\Big|_{D_\mu \to n\cdot D\frac{\bar{n}^\mu}{2} + \bar{n}\cdot D_c\frac{n^\mu}{2} + D^\mu_{c\perp}}$$

I have only written the Lagrangian for one collinear sector, but in our applications we'll always have two sectors, $p_{c_1} \sim (\lambda^2, 1, \lambda)$ & $p_{c_2} \sim (1, \lambda^2, \lambda)$. The second collinear sector is obtained by replacing $n^\mu \leftrightarrow \bar{n}^\mu$ (which implies $x_+ \leftrightarrow x_-$) in the first sector.

Let us now discuss gauge invariance. In the same way we have expanded the Lagrangian, we will also expand the transformations. Furthermore, we will consider both soft and collinear gauge transformations.

Since the collinear transformations involve a field with large energy, the soft fields cannot transform under them: $\quad V_c = \exp[i\alpha^a t^a]$

$$\xi_c \to V_c \xi_c \qquad\qquad q_s \to q_s$$

$$A_s^\mu \to A_s^\mu$$

$$A_{c\perp}^\mu \to V_c A_{c\perp}^\mu V_c^\dagger + \frac{i}{g} V_c \overset{= (\partial_\perp^\mu V_c^\dagger)}{[\partial_\perp^\mu , V_c^\dagger]}$$

$$\bar{n}A_c \to V_c \bar{n}A_c V_c^\dagger + \frac{i}{g} V_c [\bar{n}\partial, V_c^\dagger]$$

$$n\cdot A_c \to V_c nA_c V_c^\dagger + \frac{i}{g} V_c [n\cdot D_s(\underset{\Uparrow\,!}{x_-}), V_c^\dagger]$$

The last transformation law is special. It ensures that $in\cdot D = in\partial + gA_c(x) + gA_s(x_-)$ transforms as $V\, in\cdot D\, V^\dagger$.

Let's now consider soft transformations $V_s = \exp[i\alpha_s^a t^a]$.

The soft fields transform in the standard way

$$A_s^\mu \to V A_s^\mu V^+ + \frac{i}{g} V[\partial^\mu, V^+]$$

$$q_s \to V q_s$$

When transforming collinear fields, we must expand $\phi_c \phi_s \to \phi_c(x)\phi_s(x_-)$, therefore

$$\xi \to V_s(x_-)\xi$$

$$A_c^\mu \to V_s(x_-) A_c^\mu V_s^+(x_-)$$

Note that the "missing" $V_s[\partial^\mu, V^+]$ term is power suppressed for $A_{c\perp}^\mu$ and $\bar{n}\cdot A_c$. The small component of the collinear field, on the other hand, appears in the combination

$$n\cdot A_c(x) + n\cdot A_s(x_-)$$

$$\to V_s(x_-)\left[n\cdot A_c(x) + n\cdot A_s(x_-)\right]V_s(x_-)$$
$$+ \frac{i}{g} V_s(x_-)\left[n\cdot\partial, V_s(x_-)\right]$$

and transforms as expected.

It is easy to check that our Lagrangian is separately invariant under soft and collinear gauge transformations. The different covariant derivatives all transform as $V\, iD_\mu V^\dagger$ and the fermions as Vq and $V\xi$, with $x \to x_-$ in the appropriate places.