7.2. Scalar SCET

In the last lecture, we have expanded a triangle integral around the limit where the in- and outgoing particles were energetic $p_1^2 \ll E_1^2$, $p_2^2 \ll E_2^2$ [or, to say it in a Lorentz invariant, way $p_1^2 \sim p_2^2 \ll (p_1-p_2)^2$]

We will now construct an effective theory, which implements this expansion on the level of the Lagrangian. Calculating with the leading-order Lagrangian, one automatically obtains the leading order expansion of the full-theory results. Adding also subleading terms, one can systematically also obtain higher-order corrections.

The structure of the effective theory relates very closely to the expansion in the strategy of regions method. The correspondence is basically one-to-one: the Feynman rules in Soft-Collinear Effective Theory simply produce the contributions of the different regions.

Let us briefly recapitulate what the ingredients of the strategy of region were:

- Light-cone vectors in the directions of the energetic particles $n^\mu = (1,0,0,1)$, $\bar{n}^\mu = (1,0,0,-1)$
- Expansion parameter $\lambda^2 \sim \frac{p^2}{E^2}$
- Momentum regions

	$(n\cdot k\,,\bar{n}\cdot k\,,k_\perp)$	
hard	$(1\,,1\,,1)$	$k^2 \sim 1$
coll-1	$(\lambda^2\,,1\,,\lambda)$	$k^2 \sim \lambda^2$
coll-2	$(1\,,\lambda^2\,,\lambda)$	$k^2 \sim \lambda^2$
soft	$(\lambda^2\,,\lambda^2\,,\lambda^2)$	$k^2 \sim \lambda^4$

- Expand in each region, integrate over all momenta in each case.

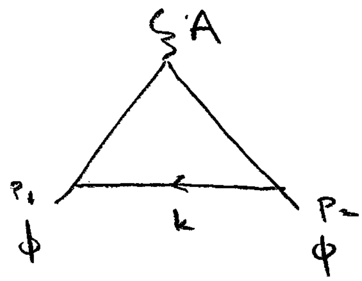
For simplicity, we'll first construct the effective theory for ϕ^3-theory: the construction is basically the same as for QCD.

The full theory Lagrangian is

External current

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{g}{3!}\phi^3 + A\phi^2 .$$

Up to a prefactor, our scalar integral corresponds to



To obtain the effective theory, we replace

$$\phi(x) \longrightarrow \phi_s(x) + \phi_{c_1}(x) + \phi_{c_2}(x)$$

soft field collinear fields

We do not introduce a hard field: we'll integrate out the hard contribution and absorb it into the Wilson coefficients (i.e. the couplings) of operators built from soft and collinear fields, in analogy to the OPE.

Splitting the fields into three modes is not good enough yet: to get the strategy of region result, we have to expand in the small momentum components.

How does the collinear field $\phi_{c_1}(x)$ know that that its momentum scales as $(\lambda^2, 1, \lambda)$?

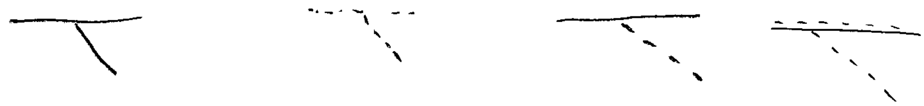
To have this scaling, we

- introduce external sources which inject the appropriate momentum.
- only allow interactions compatible with the scaling.

Expanding $\phi \to \phi_{c_1} + \phi_{c_2} + \phi_s$, we keep only the following terms:

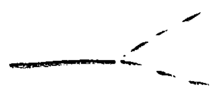
$$\mathcal{L}_{SCET} = \frac{1}{2}(\partial_\mu \phi_{c_1})^2 + \frac{1}{2}(\partial_\mu \phi_{c_2})^2 + \frac{1}{2}(\partial_\mu \phi_s)^2$$

$$- \frac{g}{3!}(\phi_{c_1}^3 + \phi_{c_2}^3 + \phi_s^3) - \frac{g}{2!}(\phi_{c_1}^2 \phi_s + \phi_{c_2}^2 \phi_s)$$



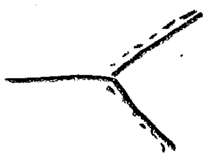
Terms forbidden by momentum conservation

$\phi_{c_1} \phi_s^2$



An energetic particle decays to two particles with small energ

$\phi_{c_1} \phi_{c_2}^2$



A particle moving fast in the $+z$ direction turns into two moving in the $-z$ direction.

For the last two terms we need to implement the expansion in small momentum. In position space, this corresponds to an expansion in derivatives.

To obtain the expansion, write the $\phi_{c_1}^2 \phi_s$ term in terms of Fourier transformed fields.

$$\delta L_{int} = \int d^4x \, \phi_{c_1}^2(x)\, \phi(x) =$$

$$\int d^4x \int_{p_1,p_2,p_3} e^{-i(p_1+p_2+p_3)x} \, \phi_{c_1}(p_1)\, \phi_{c_2}(p_2)\, \phi_s(p_3)$$

$$\begin{array}{lll} & & \quad n\cdot p \quad \bar{n}\cdot p \quad p_\perp \\ (p_1+p_2+p_3)^\mu & \text{scales} & (\lambda^2, \; 1, \; \lambda) \\ x^\mu & \text{scales} & (1, \; 1/\lambda^2, \; 1/\lambda) \\ p_s & \text{scales} & (\lambda^2, \; \lambda^2, \; \lambda^2) \\ & & \quad p_+ \; + \; p_- \; + \; p_\perp \end{array}$$

$$\Rightarrow \quad p_s \cdot x = \underset{O(1)}{p_{s+}\, x_-} + \underset{O(\lambda)}{p_{s\perp}\cdot x_\perp} + \underset{O(\lambda^2)}{p_{s-}\cdot x_+}$$

$$\delta L_{int} = \int d^4x \, \phi_{c_1}^2(x)\, \phi(x)$$

$$= \int d^4x \, \phi_{c_1}^2(x) \left[1 + x_\perp \cdot \partial_\perp + x_+ \cdot \partial_{x_-} + \dots \right] \phi_s(x) \Big|_{x=x_-}$$

$$= \int d^4x \, \phi_{c_1}^2(x)\, \phi_s(x_-) + \dots$$

The expansion in derivatives is called "multipole-expansion" in the original references (Beneke & Feldmann et al. '02, '03)

There is an alternative formulation, the "Label-formalism" by Bauer, Stewart et al. '01)

It turns out that the effective Lagrangian we have constructed is <u>exact</u>, despite the fact that we did not consider loop corrections in it's construction.

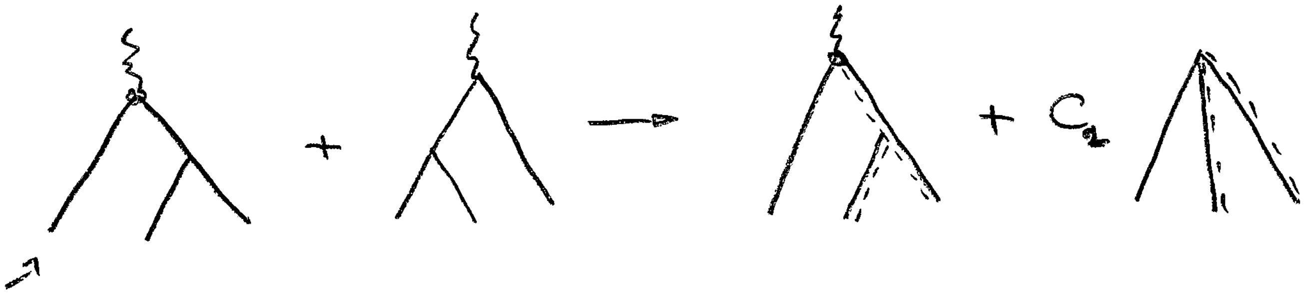
The reason is that each sector ($\mathcal{L}_s, \mathcal{L}_{c_1}, \mathcal{L}_{c_2}$) is equivalent to the full theory and all loop integrals with only $c_1 + s$ or $c_2 + s$ are scaleless and vanish.

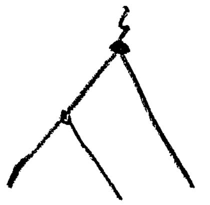
The only part with nontrivial matching is the current operator $A\phi^2$

At lowest order .

$$\phi^2 \longrightarrow i\, \phi_{c_1} \phi_{c_2}$$

At $O(g)$: $\quad \phi^2 \to \phi_{c_1}\phi_{c_2} + C_2\left(\phi_{c_1}\phi_{c_2}^2 + \phi_{c_1}^2\phi_{c_2}\right)$





$$= \frac{i}{(p_1+p_{2a})^2}\, ig \;=\; \frac{1}{2p_1^- \cdot p_{2a}^+}(-g)$$

$p_1 \quad p_{2a} \quad p_{2b}$

$\quad\; xp_2 \quad (1-x)p_2$

$$\bar{n}\cdot p_1\, n\cdot p_{2a} \sim \lambda^2$$

$$\phi^2 \to \phi_{c_1}\phi_{c_2} + g\left[\left(\frac{1}{\bar{n}\cdot\partial}\phi_{c_1}\right)\left(\frac{1}{n\partial}\phi_{c_2}\right)\phi_{c_2} + (1\leftrightarrow 2)\right]$$

$p^\mu \sim i\partial^\mu$

The appearance of an inverse derivative is at first sight disturbing. Note however that it corresponds to $\frac{1}{E}$ and by construction E is large.

An alternative way of writing the inverse derivative is as an integral

$$\frac{i}{in\partial + i\varepsilon}\phi(x) = \int_{-\infty}^{0} ds\, \phi(x + s\cdot n)$$

It is a characteristic feature of SCET that the operators are non-local along the directions of large light-cone momentum. To write down the most general SCET operator, one smears the fields along the light-cone

$$A\,\phi^2(x) \longrightarrow A\left[\mathcal{O}_2(x) + \mathcal{O}_3(x) + \cdots\right]$$

$$\mathcal{O}_2 = \int ds \int dt\; C_2(s,t)\, \phi_{c_1}(x + s\bar{n})\, \phi_{c_2}(x + tn)$$

$$\mathcal{O}_3 = \int ds \int dt_1 \int dt_2\; C_3(s,t_1,t_2)\, \phi_{c_1}(x + s\bar{n}) \times \phi_{c_2}(x + t_1\bar{n})\, \phi_{c_2}(x + t_2\bar{n}) + (1 \leftrightarrow 2)$$

We found $C_2(s,t) = \delta(s)\,\delta(t) + O(g^2)$

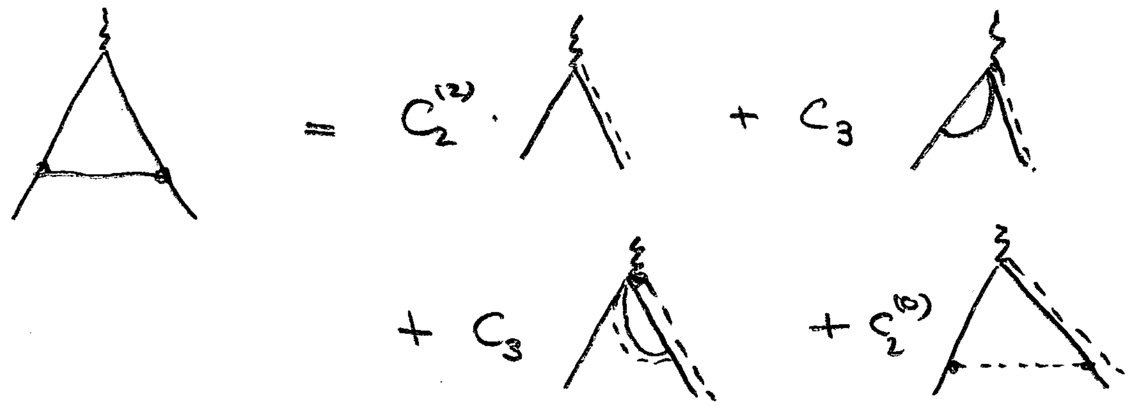
$$C_3(s,t_1,t_2) = g\,\Theta(-s)\,\Theta(-t_1)\,\delta(t_2) + O(g^3)$$

The dependence on s,t is equivalent to dependence on the large energies in momentum space

$$\delta(s) \leftrightarrow 1$$

$$\Theta(-s) \leftrightarrow \frac{1}{E_1}$$

Let us now check how the calculation of the ϕ^3 vertex diagram looks in the effective theory



$C_2^{(0)}$ is the tree-level Wilson coefficient of O_2

$C_2^{(2)}$ it's one-loop value.

The four diagrams are in one-to-one contribution to the hard, coll-1, coll-2, and soft contributions we encountred when we did the strategy of region expansion earlier.

For order-by-order calculations, the strategy of regions is more efficient. SCET is useful to derive all-order properties such as factorization theorems. Furthermore the RG in the effective theory can be used to resum log's of $\frac{p^2}{E^2}$ to all orders.

7.2.1 Factorization of the Sudakov form factor

Let us use our scalar SCET for a factorization discussion. The four-dimensional theory has a dimensionful coupling, which complicates the analysis. Let's instead consider the theory in six dimensions.

$$S = \int d^dx \left[\frac{1}{2}(\partial_\mu \phi)^2 - g\phi^3\right]$$

$$\Downarrow$$

$$\rightarrow [\phi] = \frac{d-2}{2} \qquad [\phi] = 1 \text{ in } d=4$$

$$[\phi] = 2 \text{ in } d=6$$

$$\rightarrow [g] = 1 \text{ in } d=4$$

$$[g] = 0 \text{ in } d=6 \qquad \left(3\cdot\frac{d-2}{2} - d = \frac{d-6}{2}\right)$$

Now let's establish the power counting in the EFT, i.e. see how the various fields count:

$$\langle \phi_c(x)\,\phi_c(0)\rangle \sim \int d^6p \underbrace{e^{-ipx}}_{1} \frac{i}{p^2}$$

$$\sim \lambda^2 \cdot \underbrace{\lambda^4}_{\text{4 transverse directions.}} \cdot \frac{1}{\lambda^2} = \lambda^4$$

$$\phi_c \sim \lambda^2$$

$$\int d^6p = \frac{1}{2}\int dn\cdot p \int d\bar{n}\cdot p \int d^4p_\perp$$

$$\langle \phi_s(x)\, \phi_s(0) \rangle = \int d^6p\, e^{-ipx} \frac{i}{p^2} \sim (\lambda^2)^6 \cdot \frac{1}{\lambda^4} \sim \lambda^8$$

$$\phi_s \sim \lambda^4$$

Now let's look at <u>terms in L</u>

$$\int d^6x\, \frac{1}{2} (\partial_\mu \phi_c)^2 \sim \frac{1}{\lambda^2} \frac{1}{\lambda^4} \lambda^2 (\lambda^2)^2 = \lambda^0 \quad \checkmark$$

$$\int d^6x\, \frac{1}{2} (\partial_\mu \phi_s) \sim \frac{1}{(\lambda^2)^6} \lambda^4 (\lambda^4)^2 = \lambda^0$$

$$\int d^6x\, g\, \phi_c^3 \sim \frac{1}{\lambda^6} (\lambda^2)^3 = \lambda^0$$

$$\int d^6x\, g\, \phi_s^3 \sim \frac{1}{\lambda^{12}} (\lambda^4)^3 = \lambda^0$$

$$\int d^6x\, g\, \phi_c^2 \phi_s \sim \frac{1}{\lambda^6} (\lambda^2)^2 \lambda^4 = \underline{\underline{\lambda^2}} \quad \text{suppressed}$$

x scales collinear

<u>Current operators</u>

$$\int d^6x\, A\, \phi_{c_1} \phi_{c_2} \sim 1$$

$$\frac{1}{\lambda^4} \quad \lambda^2 \quad \lambda^2$$

$$\int d^6x\, A\, (\phi_{c_1})^2 \phi_{c_2} \sim \lambda^2$$

$$\frac{1}{\lambda^4} \quad (\lambda^2)^2 \quad \lambda^2$$

$$\int d^6x\, A\, \phi_{c_1} \phi_{c_2} \phi_s \sim \lambda^4$$

In summary:

$$\int d^6x \, \mathcal{L}_{SCET} = \int d^6x \left[\mathcal{L}_{c_1} + \mathcal{L}_{c_2} + \mathcal{L}_s \right] + O(\lambda^2)$$

Current operator

$$\int d^6x \, A \, \phi^2(x) \sim \int d^6x \int ds \int dt \, C(s,t) \, \phi_{c_1}(x+s\bar{n}) \, \phi_{c_2}(x+t\bar{n})$$
$$+ O(\lambda^2)$$

Since soft-collinear interactions are power suppressed, we now obtain a factorization theorem

$$G(p_1, p_2) = \int d^6x_1 \int d^6x_2 \, e^{-ip_1x_1 + ip_2x_2}$$
$$\langle 0 | T\{ \phi(x_1) \, A\phi^2(0) \, \phi(x_2) \} | 0 \rangle$$

$$= \int d^6x_1 \int d^6x_2 \, e^{-ip_1x_1 + ip_2x_2} \int ds \int dt \, C(s,t)$$
$$A \, \langle 0 | T\{ \phi_{c_1}(x_1) \, \phi_{c_1}(s\bar{n}) \} | 0 \rangle \langle 0 | T\{ \phi_{c_2}(t\bar{n}) \phi(x_2) \} | 0 \rangle$$

Use translation invariance

$$\langle 0| T\{ \phi_{c_1}(x)\, \phi_{c_2}(s\bar{n}) \} |0\rangle = \langle 0| T\{ \phi_{c_1}(x - s\bar{n})\, \phi_{c}(0) \} |0\rangle$$

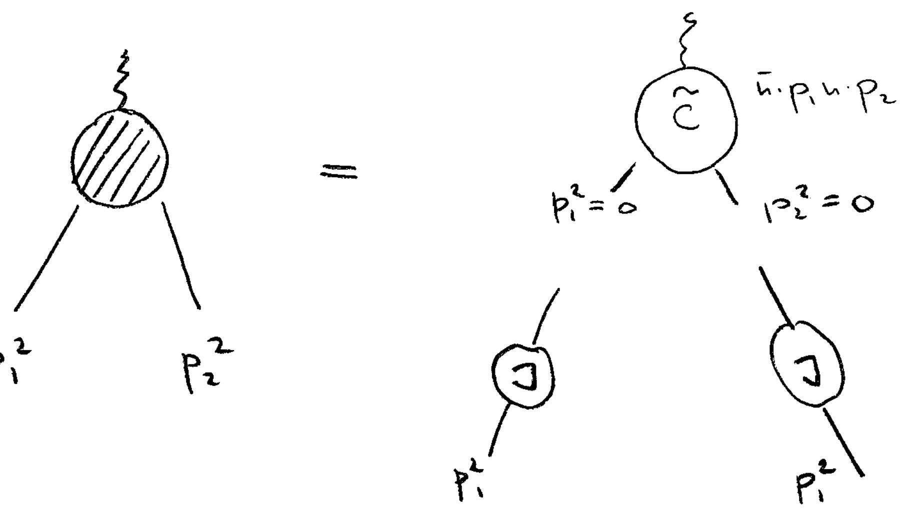
and translate $x_1 \to x_1 + s\bar{n}$, $x_2 \to x_2 - t\bar{n}$

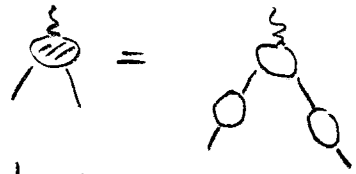
$$G(p_1, p_2) = \int ds \int dt\; C(s,t)\, e^{is p_1 \cdot \bar{n}}\, e^{-it p_2 \cdot n}\; J(p_1^2)\, J(p_2^2)$$

$$\left[J(p_1^2) = \int d^6x\, e^{-ip_1 x} \langle 0| T\{ \phi_{c_1}(x)\, \phi_{c_1}(0) \} |0\rangle \right.$$

$$\boxed{G(p_1 p_2) = \tilde{C}(\bar{n}\cdot p_1\, n \cdot p_2)\, J(p_1^2)\, J(p_2^2)}$$

We have factorized the Green's function into a product of a hard function $\tilde{C}$ and two jet-functions J. Note that $L_{c_1} \equiv L_{\phi^3}$, so we can calculate J using the full theory.



Note that one can always write  , what makes our theorem nontrivial is that the form-factor part  is evaluated for $p_1^2 = p_2^2 = 0$. We have factorized the form-factor into a high-energy part $\tilde{C}(\bar{n}p_1, n p_2)$ and a low energy contribution encoded in $J(p_1^2)$ and $J(p_2^2)$

It would be fun to use the theorem to resum Sudakov logarithms $\alpha_s^n \ln^{2n}\left(\frac{p_1^2 p_2^2}{Q^4}\right)$ to all orders using the renormalization group in the effective theory, but we will move on to QCD.