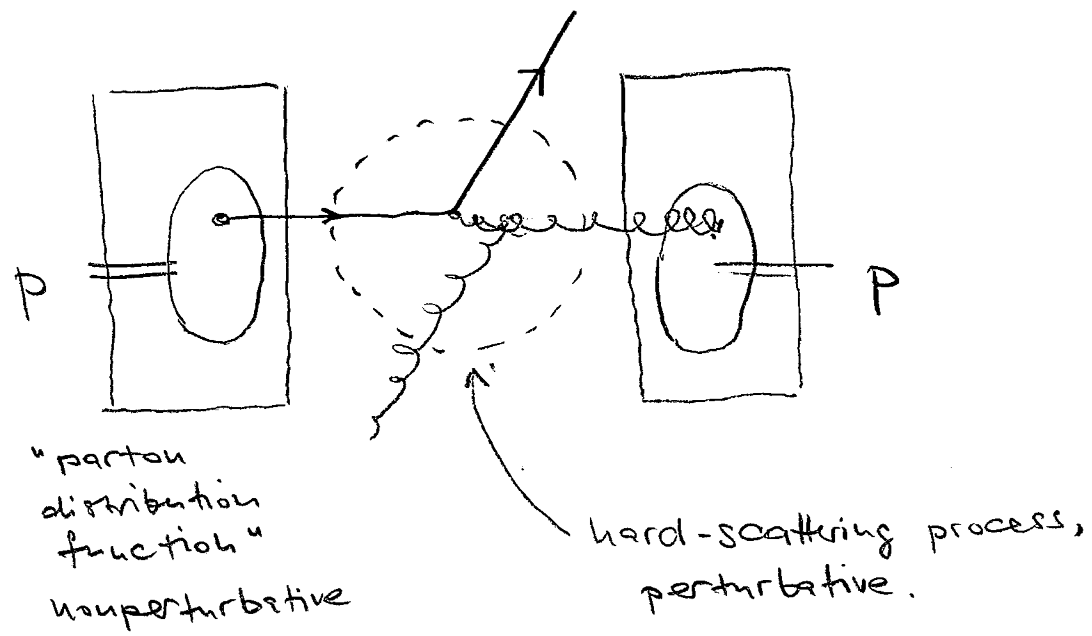
7. Soft-Collinear Effective Theory

We have defined observables such as event shapes and jet rates which are insensitive to low energy QCD effects from soft and collinear emissions.

However in hadronic collisions, low energy QCD effects are unavoidable: the scattering process involves a bound state of quarks and gluons, which cannot be treated perturbatively.

What saves the day are factorization theorems.



The parton distribution functions (PDFs) give the probability* to find a quark or gluon with momentum $x \cdot P$ inside a hadron with momentum P.

The cross section factorizes as

$$\sigma^{had}(P_1,P_2) = \underbrace{\sum_{a,b}}_{\{a,b\} = \{g,u,d,s,\ldots,\bar{u},\bar{d},\bar{s},\ldots\}} \int_0^1 dx_1 \int_0^1 dx_2 \underbrace{f_{a/H_1}(x_1)\, f_{b/H_2}(x_2)}_{\text{PDFs}}\, \underbrace{\sigma^{ab}}_{\substack{\text{partonic}\\ \text{cross-section}}}(x_1P_1, x_2P_2)$$

The proof of such factorization theorems is quite involved, so most (all?) books avoid the subject. Using modern effective theory methods makes such proofs much simpler and more transparent. In the following we'll develop the necessary formalism. It will take some effort to construct SCET, but once we're done the factorization proofs will be relatively straightforward.

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*.) Roughly speaking. They are renormalized which makes the interpretation less straightforward.

7.1. Strategy of regions

The "strategy of regions" (Beneke and Smirnov '97) is a very general and efficient method to expand loop integrals around various limits. The expansion is obtained by splitting the integral into contributions from different regions. In our case, the regions will be regions of soft and collinear momentum, but let's first warm up with a 1-d example integral

$$I = \int_0^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{\ln\left(\frac{M}{m}\right)}{M^2 - m^2}$$

$$= \ln\left(\frac{M}{m}\right)\frac{1}{M^2}\left\{1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots\right\} \quad \text{for } m^2 \ll M^2$$

Can we expand on the level of the integrand? No:

$$I \overset{?}{=} \int_0^\infty dk \frac{k}{(k^2+M^2)} \frac{1}{k^2}\left(1 - \frac{m^2}{k^2} + \dots\right)$$

IR - divergence!

Problem: In the region $k \sim m$, the expansion of the integrand is not justified!

Solution: Split the integration in two regions

$$m \ll \Lambda \ll M.$$

$$I = \Big(\underbrace{\int_0^\Lambda dk}_{(I)} + \underbrace{\int_\Lambda^\infty dk}_{(II)} \Big) \frac{k}{(k^2+m^2)(k^2+M^2)}$$

In (I) $k \sim m \ll M$, expand

$$\frac{1}{(k^2+m^2)(k^2+M^2)} = \frac{1}{(k^2+m^2)} \frac{1}{M^2} \left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \dots \right)$$

In (II) $m \ll k \sim M$, expand

$$\frac{1}{(k^2+M^2)} \frac{1}{k^2} \left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} - \dots \right)$$

$$I_{(I)} = -\frac{\Lambda^2}{2M^4} - \frac{1}{M^2} \ln\left(\frac{m}{\Lambda}\right) + \dots$$

$$I_{(II)} = +\frac{\Lambda^2}{2M^4} - \frac{1}{M^2} \ln\left(\frac{\Lambda}{M}\right) + \dots$$

In the sum

$$I = I_{(I)} + I_{(II)} = - \frac{1}{M^2} \ln\left(\frac{m}{M}\right) \checkmark$$

the dependence on the separator Λ cancels, and we indeed reproduce the first term in the expansion of the integral.

Note that there is a close correspondence between what we did and the concept of an effective theory. We have split the integral in a low energy region (I) and a high energy region (II). Λ is the the UV cut-off in the low energy region.

While our method works fine, it is very hard to evaluate loop integrals in cut-off regularization. It turns out that we can get the same result using dimensional regularization instead of a hard cut-off.

Consider the integral in dimensional regularization

$$I = \int dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)(k^2+M^2)}$$

⌜This simplified version is good enough; our integral is finite and we only want $\varepsilon \to 0$ at the end of the day.⌟

Now calculate the contributions of (I) & (II), but <u>without</u> a cut-off.

$$I_{(I)} = \int_0^\infty dk \frac{k^{-\varepsilon}}{(k^2+m^2)} \frac{1}{M^2} \left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \dots \right)$$

$$= + \frac{1}{M^2} \left[+\frac{1}{\varepsilon} - \ln(m) \right]$$

UV divergence "$\varepsilon > 0$"

$$I_{(II)} = \int_0^\infty dk \frac{k^{-\varepsilon}}{(k^2+M^2)} \frac{1}{k^2} \left(1 - \frac{m^2}{k^2} + \dots \right)$$

$$= + \frac{1}{M^2} \left[-\frac{1}{\varepsilon} + \ln(M) \right]$$

IR divergence "$\varepsilon < 0$"

The sum is

$$I = I_{(I)} + I_{(II)} = -\frac{1}{M^2} \ln\left(\frac{m}{M}\right) \checkmark$$

The $\frac{1}{\varepsilon}$ divergences have cancelled.

That the procedure works is surprising at first sight. It looks like we are double counting by integrating over the full phase-space in both (I) & (II).

One way to see that this is not the case is to subtract from our low energy integral it's expansion around high energies

$$I'_{(I)} = \int_0^\infty dk \, \frac{k^{-\varepsilon}}{M^2} \left[\frac{1}{k^2+m^2} - \left(\frac{1}{k^2} - \frac{m^2}{k^4} + \frac{m^4}{k^6} \right) \right]$$

After the subtraction, the integrand $I'_{(I)}$ is no longer sensitive to the region of large k, since the integrand goes like $\frac{1}{k^8}$ for large k.

However, all subtraction terms are scaleless and vanish in dim. reg.

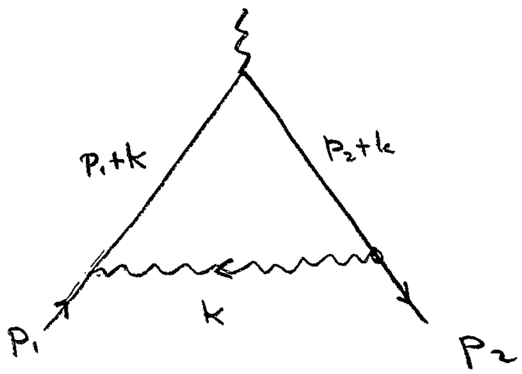
So $I'_{(I)} = I_{(I)}$.

$\rightsquigarrow$ The overlap between regions corresponds to scaleless integrals.

For applications, such as the expansion of Euclidean integrals in small or large masses, it was proven that the above procedure leads to the correct result also for n-loop integrals. Such a proof is however still missing for the application we will now consider.

7.1.1. Sudakov problem

Let us now look at the expansion in a situation where particles have large energy but small invariant masses p^2 and expand in this limit. The simplest example is the integral



$$p_1^2 \sim p_2^2 \ll (p_1-p_2)^2 = s$$

For simplicity, let's just consider the scalar integral

$$\int d^dk \frac{1}{(p_1+k)^2-i\varepsilon} \frac{1}{(p_2+k)^2-i\varepsilon} \frac{1}{k^2-i\varepsilon} = \overbrace{-i\pi^{d/2} \frac{e^{-\gamma_E \varepsilon}}{s}}^{C} V(p_1^2, p_2^2, s)$$

To perform the expansion, it is useful to introduce light-like reference vectors in the directions of the particles.

Assume that the incoming particle flys in the $+z$ direction, and p_2 is in the $-z$ direction.

Introduce $n^\mu = (1,0,0,1)$

$$\bar{n}^\mu = (1,0,0,-1)$$

$$n^2 = \bar{n}^2 = 0. \quad n \cdot \bar{n} = 2.$$

Any four-vector p^μ can be written as

$$p^\mu = (n \cdot p)\frac{\bar{n}^\mu}{2} + (\bar{n} \cdot p)\frac{n^\mu}{2} + p_\perp^\mu$$

$$= p_+^\mu + p_-^\mu + p_\perp^\mu .$$

Where $p_\perp \cdot n = p_\perp \cdot \bar{n} = 0$.

Note that $p^2 = 2 \cdot n \cdot p \, \bar{n} p \, \frac{n \cdot \bar{n}}{4} + p_\perp^2$

$$= n \cdot p \, \bar{n} p + p_\perp^2 .$$

Since p_1 flys in the z-direction $\bar{n} \cdot p_1$ is large, but since p_1^2 is small $p_\perp^\mu$ and $n \cdot p$ must be small.

More precisely, the components scale as

$$(n\cdot p,\ \bar{n}\cdot p,\ p_\perp)$$

$$p_1^\mu \sim E\,(\lambda^2, 1, \lambda) \qquad \text{with } \lambda^2 \sim \frac{p^2}{E^2}$$

$$p_2^\mu \sim E\,(1, \lambda^2, \lambda)$$

To expand the loop integral we need to consider the following regions of the loop-momentum k^μ:

hard: $k^\mu \sim (1, 1, 1)$

1-collinear: $k^\mu \sim p_1^\mu \sim (\lambda^2, 1, \lambda)$

2-collinear: $k^\mu \sim p_2^\mu \sim (1, \lambda^2, \lambda)$

soft* : $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)$

Note: any other scaling $k^\mu \sim (\lambda^a, \lambda^b, \lambda^c)$ leads to scaleless integrals after expanding.

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* This is often called ultra-soft to distinguish it from $(\lambda, \lambda, \lambda)$.

So let's look at the contributions from the different regions:

<u>hard:</u>

$$\int d^dk \frac{1}{(2p_1\cdot k + k^2)(2p_2\cdot k + k^2)\,k^2} + \ldots = +C\cdot V_h$$

$$V_h = \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln(-s) + \frac{1}{2}\ln^2(s) - \frac{\pi^2}{12}$$

<u>1-collinear</u>

$$\int d^dk \frac{1}{(p_1+k)^2}\,\frac{1}{\underbrace{(2p_{2+}k + i\varepsilon)}_{O(\lambda^0)}}\,\frac{1}{k^2} = +C\cdot V_{c_1}$$

$$V_{c_1} = -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln(-p_1^2) - \frac{1}{2}\ln^2(-p_1^2) + \frac{\pi^2}{12}$$

<u>soft:</u>

$$\int d^dk \frac{1}{(2p_{1-}k_+ + p_1^2)(2p_{2+}k_- + p_2^2)\,k^2} = c\cdot V_s$$

$$= \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\left(\frac{(-p_1^2)(-p_2^2)}{s}\right) + \frac{1}{8}\ln^2\left(\frac{(-p_1^2)(-p_2^2)}{s}\right) + \frac{\pi^2}{4}$$

In the sum of all terms the divergences cancel

and one obtains:

$$V = V_h + V_{c_1} + V_{c_2} + V_s = \underline{\frac{1}{4} \ln^2\left(\frac{(-p_1^2)(-p_2^2)}{s^2}\right)}$$

Let's calculate one of these integrals explicitly:

$$c\, V_s = \int d^qk \frac{1}{(2p_{1-}\cdot k_+ + p_1^2)(2p_{2+}\cdot k_- + p_2^2)\, k^2}$$

$$\left[\int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \frac{2}{(a + b\eta_1 + c\eta_2)^3} = \frac{1}{abc} \right.$$

$$= \int d^qk \int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \frac{2}{\left[(k + \eta_1 p_{1-} + \eta_2 p_{2+})^2 - (\eta_1 p_{1-} + p_{2+}\cdot\eta_2)^2 - \eta_1 p_1^2 - \eta_2 p_2^2\right]^3}$$

$$= -i\pi^{d/2} \frac{\Gamma(3 - d/2)}{2} \cdot 2 \int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \left[\eta_1 p_1^2 + \eta_2 p_2^2 + 2\eta_1\eta_2 \overbrace{p_{1-}\cdot p_{2+}}^{\approx s}\right]^{d/2 - 3}$$

$$= -i\pi^{d/2}\,\Gamma\left(3-\frac{d}{2}\right)\frac{1}{p_1^2 p_2^2}\int_0^\infty d\eta_1\int_0^\infty d\eta_2\,\left[\eta_1+\eta_2+\eta_1\eta_2\cdot a\right]^{-1-\varepsilon}$$

with $a = \dfrac{s}{(-p_1^2)(-p_2^2)}$

$$\eta_1^{-1-\varepsilon}\left[1+\eta_2+\eta_1\eta_2\cdot a\right]^{-1-\varepsilon}$$

$$= -i\pi^{d/2}\,\Gamma(1+\varepsilon)\frac{1}{p_1^2 p_2^2}\int_0^\infty d\eta_1\int_0^\infty d\eta_2\,\eta_1\left[\eta_1+\eta_1\eta_2+\eta_1^2\eta_2\cdot a\right]^{-1-\varepsilon}$$

$$= -i\pi^{d/2}\,\Gamma(1+\varepsilon)\frac{1}{p_1^2 p_2^2}\underbrace{\int_0^\infty d\eta_1\,\frac{\eta_1^{-1-\varepsilon}}{1+\eta_1}}_{\Gamma(1-\varepsilon)\Gamma(\varepsilon)}\underbrace{\int_0^\infty d\eta_2\,\left[1+\eta_2\right]^{-1-\varepsilon}}_{1/\varepsilon}$$

$$= -i\pi^{d/2}\,\Gamma(1-\varepsilon)\Gamma(\varepsilon)^2)\frac{1}{s}\left(\frac{(-p_1^2)(-p_2)^2}{s}\right)^{-\varepsilon}$$

$$\rightsquigarrow V_h = \left(\frac{1}{\varepsilon^2}+\frac{\pi^2}{4}\right)\left(\frac{(-p_1^2)(-p_1^2)}{s}\right)^{-\varepsilon}$$