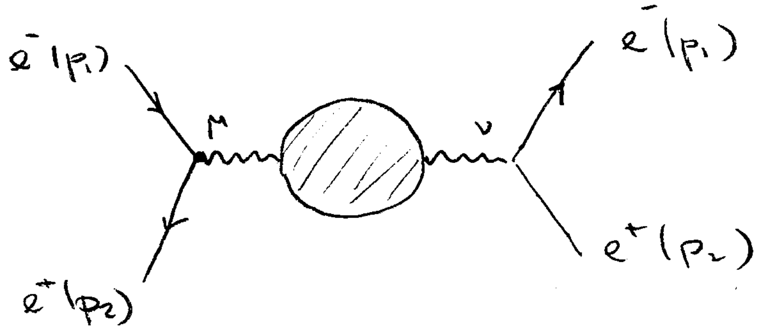
5.2. Operator analysis of e^+e^-

The diagrams for $e^+e^- \to e^+e^-$ whose imaginary part gives $e^+e^- \to$ hadrons have the form



The amplitude is

$$i\mathcal{M} = (-ie)^2 \; \bar{v}(p_2)\gamma^\mu u(p_1) \; \bar{u}(p_1)\gamma^\nu v(p_2)$$
$$\frac{-i}{s}\, i\Pi^h_{\mu\nu}(q)\, \frac{-i}{s}$$

$\Pi_{\mu\nu}$ is the hadronic part of the self energy and has the form

$$\Pi^{\mu\nu}_h(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi_h(q^2)$$

[This follows from the Ward identity $q^\mu \Pi^{\mu\nu} = 0$]

In our case only the $g^{\mu\nu}$-part contributes.

So $\quad i\mathcal{M} = e^2 \frac{1}{s^2}\, i\Pi(q^2)\, \underline{\underline{s}}\, \bar{v}(p_2)\gamma_\mu u(p_1)\, \bar{u}(p_1)\gamma^\mu v(p_2)$

Now average over incoming spins, sum over outgoing

$$\frac{1}{4}\sum \bar{u}(p_1)\gamma^\mu v(p_2)\,\bar{v}(p_2)\gamma^\mu u(p_1)$$

$$= \frac{1}{4}\,\mathrm{tr}\left[\not{p}_1 \gamma^\mu \not{p}_2 \gamma^\mu\right]$$

$$= \frac{1}{4}\Big(-\,\mathrm{tr}(\not{p}_1\not{p}_2 \underbrace{\gamma^\mu\gamma_\mu}_{4}) + 2p_{2\mu}\,\mathrm{tr}[\not{p}_1\gamma^\mu]\Big)$$

$$= \frac{1}{4}(-2)\,4\,p_1\cdot p_2 = -2s/2 = -s$$

$$s = (p_1+p_2)^2 = 2p_1\cdot p_2$$

$$\Rightarrow \quad \sigma(e^+e^- \to \text{hadrons}) = -\frac{4\pi\alpha}{s}\,\mathrm{Im}\,\Pi_h(s)$$

As a check, we should now evaluate the photon self-energy.

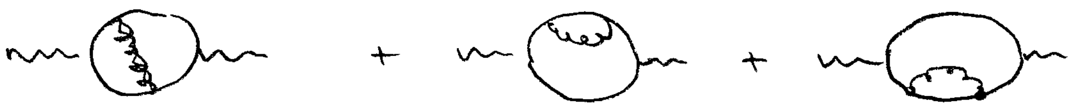
one finds $\quad \mathrm{Im}\,\Pi(s+i\varepsilon) = -\frac{\alpha}{3} N_c \sum_q e_q^2$

$$\Rightarrow \quad \sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{3s} N_c \sum_q e_q^2 \quad ✓$$

$$\to \Pi(s) = -\frac{\alpha}{3} N_c \sum_q e_q^2\left(-\frac{1}{\pi}\ln(-s-i\varepsilon)\right)$$

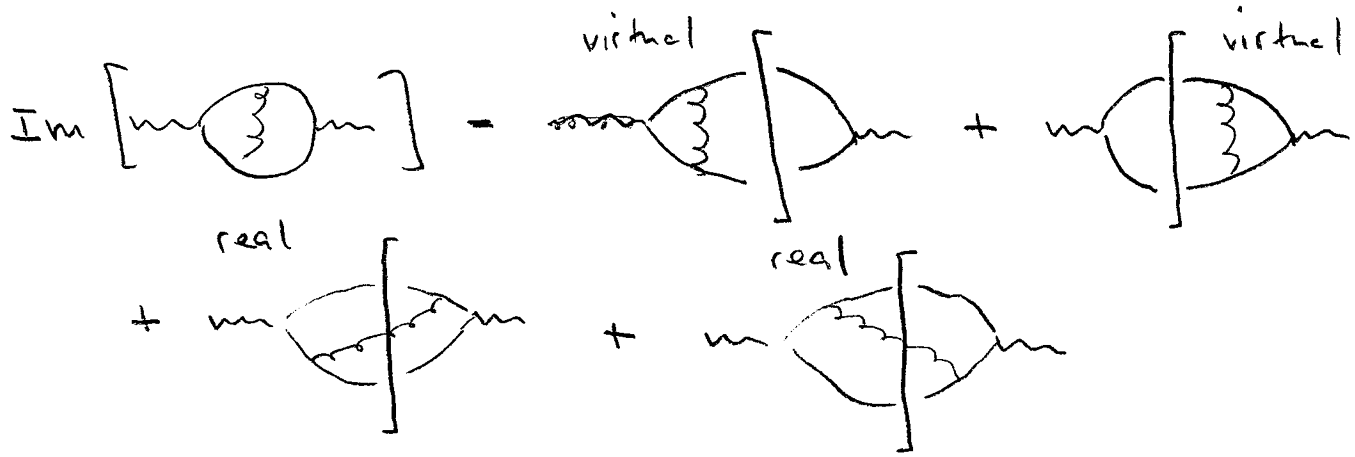
So we now get the cross section from a loop calculation <u>alone</u>.

At $O(\alpha_s)$



Since off-shell Green's functions are IR finite, we have proven to all orders in α_s that the total cross section is finite!

The cancellation between real & virtual is manifest:



(The imaginary part arises when some particles in the loop go on the mass shell.)

The electromagnetic coupling of quarks has the form

$$J^\mu = \sum_q e_q \, \bar{q}_f \gamma^\mu q_f$$

so the self-energy is

$$i\Pi_h^{\mu\nu}(q) = (ie)^2 \int d^4x \, e^{iqx} \langle 0 | T\{ J^\mu(x) J^\nu(0) \} | 0 \rangle$$

In the limit $x \to 0$, we can expand the product of currents:

$$J_\mu(x) J_\nu(0) = C^{1}_{\mu\nu}(x) + C^{q\bar{q}}_{\mu\nu}(x) m_q \bar{q} q$$
$$+ C^{G^2}_{\mu\nu}(x) (G^a_{\mu\nu})^2$$

$$C^1_{\mu\nu} \sim (x^2)^{-3} \qquad C^{q\bar{q}}_{\mu\nu} \sim (x^2)^{-1} \qquad C^{G^2}_{\mu\nu} \sim (x^2)^{-1}$$

If the Fourier transform is dominated by $x^2 \approx 0$, we get inserted by hand ...

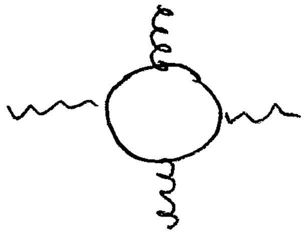
$$i\Pi^{\mu\nu}_{h}(q) = -ie^2\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$
$$\left[C^1(q^2)\,\mathbb{1} + C^{q\bar q}(q^2)\, m\bar q q \right.$$
$$\left. + \; C^{G^2}(q^2)\,(G^a_{\mu\nu})^2 \right]$$

$$C^1 \sim (q^2)^0 \qquad \underbrace{C^{q\bar q} \sim \frac{1}{q^2} \qquad C^{G^2} \sim \frac{1}{q^2}}_{\text{suppressed at high energy}}$$

Since the OPE is an operator relation, we can take an arbitrary matrix element to obtain the Wilson coefficients. In particular, we can work with unphysical quark and gluon states.

 $\longrightarrow$ $C^1(q^2)$

 $\longrightarrow$ $C^{q\bar q}(q^2)$

 $\longrightarrow$ $C^{G^2}(q^2)$

then, with the coefficients determined from perturbative calculations, we take the physical vacuum matrix element.

$$\Rightarrow \quad \sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s}\Big[\operatorname{Im} c^1(q^2)$$

$$+ \operatorname{Im} c^{q\bar q}(q^2)\underbrace{\langle 0|\, m\bar q q\,|0\rangle}_{\sim(0{,}3\,\text{GeV})^4 \text{ quark condensate}}$$

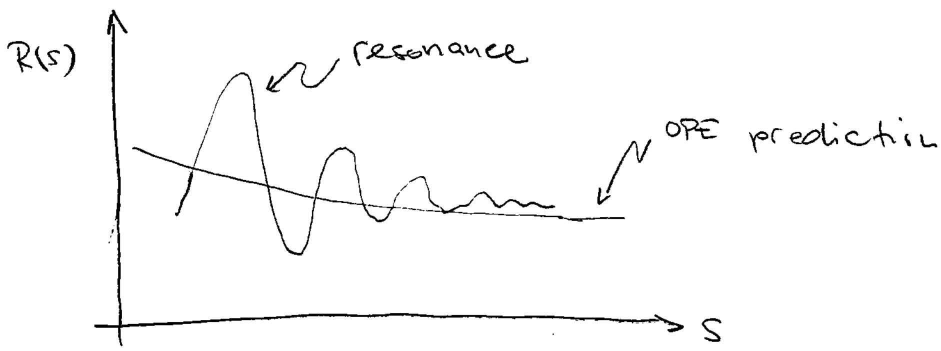
$$+ \operatorname{Im} c^{G^2}(q^2)\underbrace{\langle 0|(G^a_{\mu\nu})^2|0\rangle}_{\substack{\text{Gluon condensate}\\ \sim (0{,}5\,\text{GeV})^4}}$$

We have separated the perturbative part (Wilson coefficient) from the nonperturbative part (operator matrix elements).

So, up to corrections of order Λ_{QCD}/q^2 from the condensates, the hadron cross section is equal to the partonic cross section from perturbation theory.

There is one potential worry concerning our result. The OPE is rigorous in Euclidean space, but we are using it in Minkowski space.

At sufficiently high energy we should be fine, but at lower energies, we can hit a QCD resonance, and the OPE will break down



The reason is that near the resonance $R(s) \sim \dfrac{1}{s - M_{res}^2 + i\Gamma_{res}\Pi}$

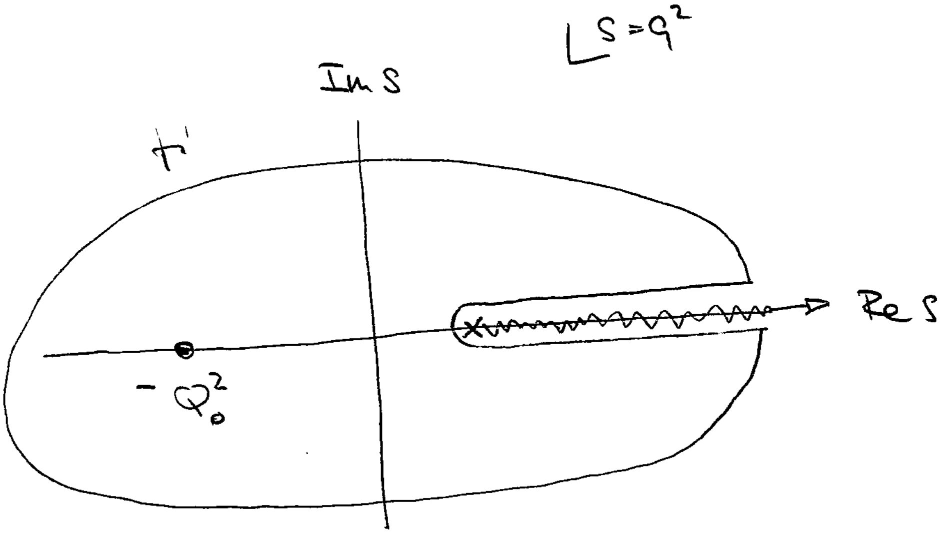
the expansion in $\frac{1}{s}$ breaks down.

It turns out that one can avoid calculating $\Pi(q^2)$ in the Minkowski region and at the same time make use of all available data.

The magic of complex analysis!

If we look at $\Pi(q^2)$ in the complex plane, we find that it is analytic everywhere, except for a cut from $s = (2m_\pi)^2 \ldots \infty$.

[In perturbation theory, the cut starts at $s = (2m_q)^2$.]



The integral $\frac{1}{2\pi i}\oint_\Gamma ds\, \Pi(s) = 0$ vanishes.

Now consider

$$I_n = -4\pi\alpha \oint_\Gamma \frac{ds}{2\pi i} \frac{1}{(q^2 + Q_0^2)^{n+1}} \Pi(s)$$

$$= -4\pi\alpha \frac{1}{n!} \frac{d^n}{ds^n} \Pi(s) \Big|_{s = -Q_0^2}$$

At $s = -Q_0^2$ it is safe to evaluate $\Pi(s)$ since we are away from all singularities at Euclidean values of s.

There is another way to evaluate the integral

$$I_n = -4\pi\alpha \int_T \frac{ds}{2\pi i} \frac{1}{(s+Q_0^2)^{n+1}} \Pi(s)$$

$$= -4\pi\alpha \int_0^\infty \frac{ds}{2\pi} \frac{1}{(s+Q_0^2)^{n+1}} \frac{1}{i}\left[\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon)\right]$$

$$= -4\pi\alpha \int_0^\infty \frac{ds}{2\pi} \frac{1}{(s+Q_0^2)^{n+1}} 2\,\mathrm{Im}\,\Pi(s)$$

$$= \frac{1}{\pi}\int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma(s)$$

So $\Pi(s)$ at Euclidean momentum is equal to an integral over the cross section.

Now evaluate

$$\underbrace{-4\pi\alpha \frac{1}{n!} \frac{d^n}{ds^n} \Pi(s) \Big|_{s=-Q_0^2}}_{\text{using the OPE.}} = \underbrace{\frac{1}{\pi} \int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma(s)}_{\text{using data}}$$

In this way one gets one of the most precise determinations of $\alpha_s(\mu)$ and of the heavy quark masses m_c and m_b.

The above relation is called a sum rule, or dispersion relation.