5.) Operator product expansion in e^+e^-

The operator product expansion (OPE) is a very useful tool with many applications in QFT. In our case, it will allow us to show

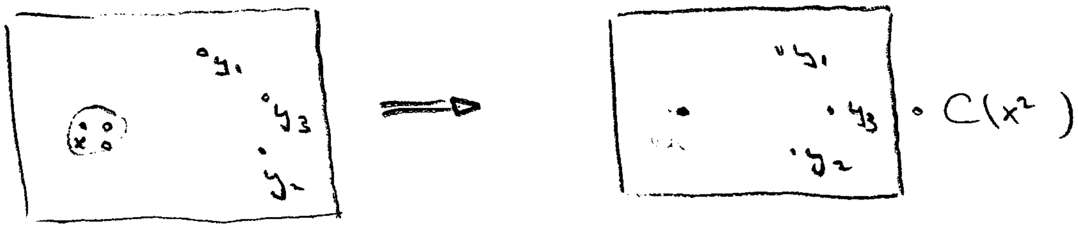
1.) that our calculation of the cross section in terms of quarks & gluons can be justified (in some cases), and

2.) what the nonperturbative corrections to our result are.

General idea: Consider product of two operators $\mathcal{O}_1(x)\,\mathcal{O}_2(0)$. Now look at Green's functions

$$G(x,0,y_1,\dots y_n) = \langle\!\langle\, \mathcal{O}_1(x)\,\mathcal{O}_2(0)\,\phi(y_1)\dots\phi(y_n)\,\rangle\!\rangle$$

In the limit $y_i^2 \gg x^2$ (in Euclidean space) we should be able to replace $\mathcal{O}_1(x)\,\mathcal{O}_2(0)$ by a local operator times a function of x^2.



Since this should hold for any Green's function, the relation should be an <u>operator</u> relation.

Wilson proposed '69

There are singularities in the limit $x \to 0$, absorb into C_n.

$$O_1^{ren}(x)\, O_2^{ren}(0) \underset{x \to 0}{=} \sum_n C_n(x^2; \mu)\, O_n^{ren}(0)$$

The operators O_n have the same quantum numbers as the product and it is useful to order the operators by their dimension.

For example:

$$\phi(x)\,\phi(0) = C_1 \cdot \mathbb{1} + C_{\phi^2}\, \phi^2(0) + C_{\phi^4}\, \phi^4 + C_{(\partial\phi)^2}\, \partial_\mu \phi(0) \partial^\mu \phi(0) + \dots$$

$\phi(x)$ has dimension of mass:

$$C_1 \sim \frac{1}{x^2} \qquad C_{\phi^2} \sim 1 \qquad C_{\phi^4} \sim x^2$$

The lowest dimensional operators have the most singular coefficients as $x \to 0$.

The higher-dim. operators are suppressed.

We can thus approximate the operator product by the first few operators on the RHS.

The naive dimensional analysis is slightly modified by renormalization, since $C_i = C_i(x^2, \mu)$ but as in the case of the coupling constant, the μ-dependence can be controlled using a RG equation. In perturbation theory higher order corrections will involve $\ln\left(\frac{x^2}{\mu^2}\right)$ terms.

5.1 The optical theorem

The optical theorem allows us to rewrite the total cross section as the imaginary part of the forward scattering amplitude. In this form, we will then apply the OPE.

It follows from the unitarity of the S-matrix, $S^{\dagger}S = \mathbb{1}$.

Write $S = 1 + iT$,

$$(1 - iT^{\dagger})(1 + iT) = 1$$

$$\to \quad \underbrace{-i(T - T^{\dagger})}_{2\,\mathrm{Im}} = T^{\dagger}T$$

Rewrite the RHS:

$$\langle p_1 p_2 | T^{\dagger}T | k_1, k_2 \rangle = \sum_X \langle p_1 p_2 | T^{\dagger} | X \rangle \langle X | T | k_1, k_2 \rangle$$

$$2\,\mathrm{Im}\,\mathcal{M}(k_1, k_2 \to p_1, p_2)$$

$$= \sum_X \mathcal{M}^*(p_1, p_2 \to p_X)\,\mathcal{M}(k_1, k_2 \to p_X)\,(2\pi)^4 \delta(k_1 + k_2 - p_X)$$

To get the standard form set $k_1 = p_1$, $k_2 = p_2$

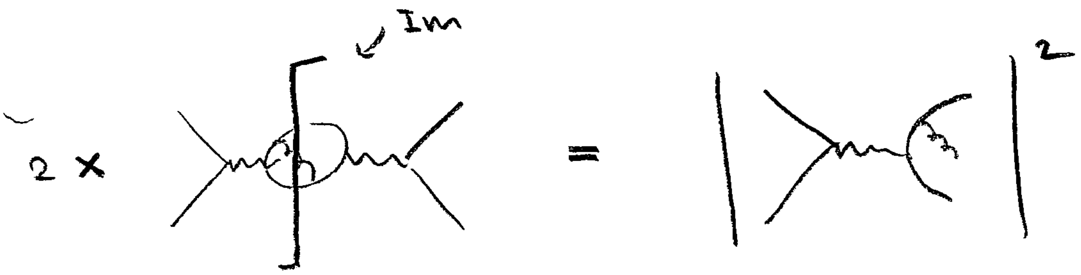
$$\Rightarrow \quad 2\,\mathrm{Im}\,\mathcal{M}(p_1, p_2 \to p_1, p_2)$$

$$= \sum_X \left|\mathcal{M}(p_1, p_2 \to p_X)\right|^2 \delta^4(p_1 + p_2 - p_X)$$

$$= 4 \underbrace{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}_{= E_{c.m.} \cdot p_{c.m.} \; = \; m_2 \cdot p_{1,lab}} \; \sigma^{tot}$$

$$\Rightarrow \quad \boxed{\mathrm{Im}\,\mathcal{M} = 2\, E_{c.m.}\, p_{c.m.}\, \sigma^{tot}}$$

Pictorially:



For e^+e^-, neglecting m_e, we have

$$\sigma_{e^+e^-} = \frac{1}{s}\,\mathrm{Im}\,\mathcal{M}(e^+e^- \to e^+e^-)$$