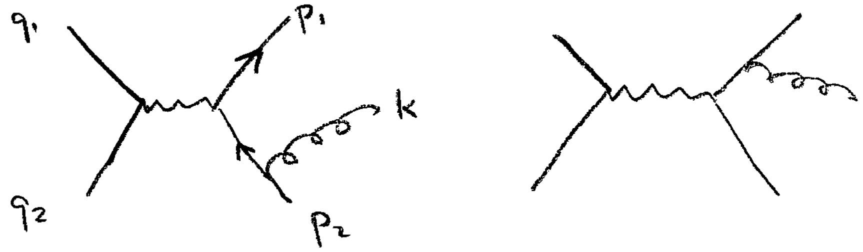
7.2. Soft and collinear divergences, IR safety

The calculation of the real emission diagrams



is straightforward (and tedious). One obtains

$$\frac{1}{4}\sum |M|^2 = 24\, C_F\, e^4 e_q^2\, g_s^2\, \frac{(p_1\cdot q_1)^2 + (p_1\cdot q_2)^2 + (p_2\cdot q_1)^2 + (p_2\cdot q_2)^2}{q_1\cdot q_2\;\; p_1\cdot k\;\; p_2\cdot k}$$

Phase space:

Let's work in the CMS and parameterize phase space as

$$q_1 = \frac{\sqrt{s}}{2}(1,0,0,1)$$

$$q_2 = \frac{\sqrt{s}}{2}(1,0,0,-1)$$

$$p_1 = x_1 \frac{\sqrt{s}}{2}(1, S_1, 0, C_1) \qquad ; \quad C_1 = \cos\theta_1, \ldots$$

$$p_2 = x_2 \frac{\sqrt{s}}{2}(1, S_2 C_\theta, S_2 S_\theta, C_2)$$

$$k = q_1 + q_2 - p_1 - p_2$$

Additional constraints:

$$x_1 = 0 \ldots 1$$

$$x_2 = 0 \ldots 1$$

$$x_3 = 2 - x_1 - x_2 = 0 \ldots 1 \quad \Rightarrow \quad x_1 + x_2 \geq 1$$

$$k^2 = 0 \quad \Rightarrow \quad C_\theta = \frac{2 - 2x_1 - 2x_2 + x_1 x_2 (1 - C_1 C_2)}{S_1 S_2 x_1 x_2}$$

Plug this into the phase-space integral, rewrite the integral as an integral over $\theta_1, \theta_2, x_1, x_2$

The angles θ_1 and θ_2 appear only in the numerator of the amplitude, since

$$k \cdot p_1 = \frac{s}{2}(1 - x_2) \qquad k \cdot p_2 = \frac{s}{2}(1 - x_1)$$

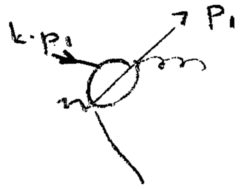
and are thus easily performed.

After lengthy algebra, one has

$$\sigma^{q\bar{q}g} = \sigma^{(0)} \int dx_1 \int dx_2 \, \frac{C_F \alpha_s}{2\pi} \, \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

The integration region is $0 \le x_1, x_2 \le 1$ $\quad x_1 + x_2 \ge 1$

Unfortunately, this is ill-defined because of the $(1-x_1)^{-1}$ & $(1-x_2)^{-1}$ factors.



$$k \cdot p_1 = \frac{s}{2}(1-x_2) = E_g E_q (1 - \cos\theta_{qg})$$

$$k \cdot p_2 = \frac{s}{2}(1-x_1) = E_g E_{\bar{q}} (1 - \cos\theta_{\bar{q}g})$$

The singularities are from the region of phase-space where particles become collinear

$$\theta_{qg} \to 0 \quad \text{or} \quad \theta_{\bar{q}g} \to 0$$

or soft, $E_g \to 0$, $E_q \to 0$ or $E_{\bar{q}} \to 0$.

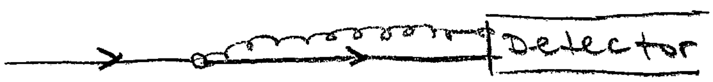
The collinear singularities are regulated (in this example) if the quark is massive

$$k \cdot p_1 = E_g (E_q - |\vec{p}_q| \cos\theta_{qg})$$

However, if the final result for the cross section would be $\propto \frac{1}{m_q}$ the perturbative results would not make sense, because $m_{u,d} \sim$ few MeV and PT no longer makes sense at such a low scale.

Furthermore, even with a quark mass, a soft singularity from $E_g \to 0$ remains.

The physical origin of these singularities is that it is not possible to distinguish a quark from a quark and a collinear gluon.



To get something sensible, we thus have to add $\sigma^{q\bar{q}} + \sigma^{q\bar{q}g}$ in the region where the gluon becomes collinear or soft.

For the total cross section we add all channels. $\sigma^{tot} = \sigma^{q\bar{q}} + \sigma^{q\bar{q}g}$, so this should be sensible. Indeed one finds that both $\sigma^{q\bar{q}g}$ and $\sigma^{q\bar{q}}$ have IR divergences at $O(\alpha_s)$ but they cancel in the total cross section

To get the cancellation one has to regularize the IR div's. To do so, one could introduce masses for quarks and gluons. A much more elegant method is to use dimensional regularization for both loop and phase-space integrals.

For the 3-body phase-space, we can use the same parameterization as before. However, angular integrations are now rewritten as

$$\int d^{d-1}p = \int dp\, p^{d-2} \underbrace{\int d\Omega_{d-1}}_{\text{solid angle}}$$

$$= \int dp\, p^{d-2} \int_{-1}^{1} d\cos\theta \, \sin^{d-4}\theta \int d\Omega_{d-2}$$

In d-dimensions, $d = 4 - 2\varepsilon$, one then finds

$$\sigma^{q\bar{q}g}(\varepsilon) = \sigma_0 \, H(\varepsilon) \int \frac{dx_1 dx_2}{P(x_1, x_2)} \frac{2\alpha_s}{3\pi} \left[\frac{(1-\varepsilon)(x_1^2 + x_2^2) + 2\varepsilon(1-x_3)}{(1-x_1)(1-x_2)} - 2\varepsilon \right]$$

$$P(x_1, x_2) = (1-x_1)^\varepsilon (1-x_2)^\varepsilon (1-x_3)^\varepsilon$$

$$x_3 = 2 - x_1 - x_2$$

$$H(\varepsilon) = \frac{3(1-\varepsilon)}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} (4\pi)^{2\varepsilon} = 1 + O(\varepsilon)$$

For $\varepsilon < 0$, the integrals can be performed and one finds

$$\sigma^{q\bar{q}g}(\varepsilon) = \sigma_0 \frac{C_F \alpha_s}{2\pi} H(\varepsilon) \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \pi^2 + O(\varepsilon) \right]$$

Two divergences: soft & collinear

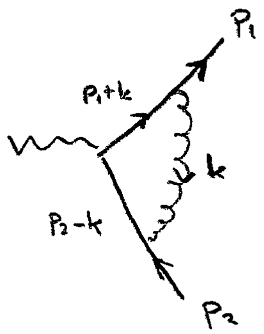
For the virtual corrections, one finds

$$\sigma^{q\bar{q}}(\varepsilon) = \sigma_0 \frac{C_F \alpha_s}{2\pi} H(\varepsilon) \left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 \right]$$

Also in the virtual part, there are collinear and soft divergences.

They arise from regions of the loop integration where the loop momentum becomes soft or collinear with the external momentum.

E.g.



involves $I = \int d^dk \, \frac{1}{(p_1+k)^2 \, (p_2-k)^2 \, k^2}$

E.g. Soft divergence:

$$\hat{\int} d^dk \, \frac{1}{2p_1\cdot k \; 2p_2\cdot k \; k^2}$$ is logarithmically IR div.

In the sum of real and virtual corrections the IR singularities cancel:

$$\sigma^{q\bar{q}}(\varepsilon) + \sigma^{q\bar{q}g}(\varepsilon) = \sigma_0 \, \frac{C_F \alpha_s}{2\pi} \cdot \left[\frac{3}{2} \right] + O(\varepsilon)$$

$$= \sigma_0 \, \frac{\alpha_s}{\pi} \, .$$

An observable in which the IR singularities present in the amplitudes cancel is called <u>infrared safe</u>.

Since exclusive cross sections, such as $\sigma^{q\bar{q}}$ or $\sigma^{q\bar{q}g}$, are all IR divergent, such observables must be inclusive.

The total cross section is as inclusive as it gets and is IR safe. Can one find other IR safe observables?

This is possible and the main categories are

1.) Jet cross sections (as defined by a jet algorithm)

2.) Event shapes

These IR safe observables must be defined in such a way that physically indistinguishable final states are always included, i.e.

If the partonic state $|X\rangle$ contributes to the cross section, then also $|X +$ "soft gluons" + "collinear gluons & quarks" $\rangle$.

We'll come back to IR safety later when we discuss event shapes & jets.