4. $e^+e^- \to$ hadrons

The simplest quantity that can be evaluated perturbatively is the total cross section

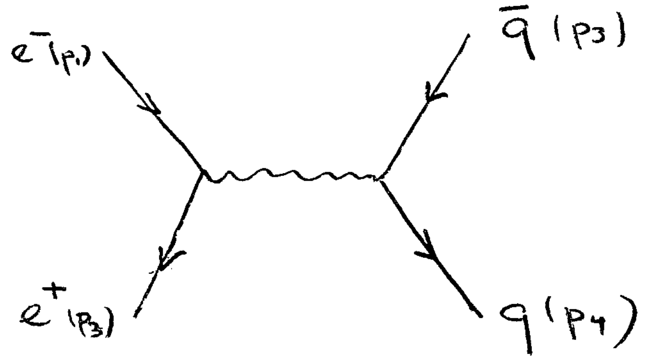
$e^+e^- \to$ hadrons.

In the following we will calculate this quantity, discuss the infrared singularities which appear at intermediate stages, and compare to data.

In chapter 5, we will then justify the use of perturbation theory.

4.1 $e^+e^- \to \bar{q}q$

To get the cross section, we evaluate the diagram



The scattering amplitude is $s = (p_1 + p_2)^2$

$$i\mathcal{M} = (\underbrace{\sqrt{Z_e}}_{\swarrow\, 1 + \frac{\alpha_s}{4\pi} C})^2 (\underbrace{\sqrt{Z_q}}_{\swarrow\, 1 + \frac{\alpha_s}{4\pi}(\ldots)})^2 \, \bar{v}(p_2, m_e)(-ie\gamma^\mu) u(p_1, m_e)$$

$$\frac{i}{s}\left(-g_{\mu\nu} + (1-\xi)(p_1+p_2)^\mu (p_1+p_2)^\nu\right)$$

$$\bar{u}(p_4, m_q)(-iee_q\gamma^\nu) v(p_3, m_q)$$

$$i\mathcal{M} = +i\frac{e^2 e_q}{s} \bar{v}(p_2, m_e)\gamma^\mu u(p_1, m_e)$$

$$* \, \bar{u}(p_4, m_q)\gamma^\mu v(p_3, m_q)$$

Now calculate $|\mathcal{M}|^2$, sum over quark spins, average over electron spins, and sum over quark colors.

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spins}} \frac{e^4 e_q^2}{s^2} \bar{v}(p_2, m_e)\gamma^\mu u(p_1, m_e)\, \bar{u}(p_1, m_e)\gamma^\nu v(p_2, m_e)$$

$$\cdot\, \bar{u}(p_4, m_q)\gamma^\mu v(p_3, m_q)\, \bar{v}(p_3, m_q)\gamma^\nu u(p_4, m_q)$$

<u>Note:</u>

$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m$$

$$\sum_s v(p,s)\bar{v}(p,s) = \not{p} - m$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4 e_q^2}{4s^2} \,\mathrm{tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_q)\gamma^\nu\right]$$

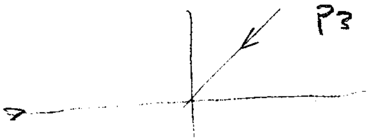
$$\circ\ \mathrm{tr}\left[(\not{p}_4 + m_q)\gamma^\mu(\not{p}_3 + m_q)\gamma^\nu\right]$$

$$= e^4 e_q^2 \left[1 + \cos^2(\theta)\right] + O(m_e^2, m_q^2)$$

c.m.s scattering angle

Γ

$$p_3 = \frac{\sqrt{s}}{2}(1, \sin\theta, 0, \cos\theta)$$



$$p_1 = \frac{\sqrt{s}}{2}(1, 0, 0, +1) \qquad p_2 = \frac{\sqrt{s}}{2}(1, 0, 0, +1)$$

$$(p_1 - p_3)^2 = -2\frac{s}{4}(1 - \cos\theta) = -\frac{s}{2}(1 - \cos\theta)$$

L

$$\delta(\sqrt{s} - E_3 - E_4)\,\delta^3(\vec{p}_3 + \vec{p}_4)$$

$$d\sigma = \int \frac{d^3p_3}{(2\pi)^3\, 2E_3} \int \frac{d^3p_4}{(2\pi)^3\, 2E_q} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$\frac{1}{4\sqrt{\underbrace{(p_1 \cdot p)^2}_{\frac{s}{2}} - m_e^4}} |\mathcal{M}|^2$$

$$= \int \frac{d^3p_3}{(2\pi)^2\, 4|\vec{p}_3|^2}\, \delta(\sqrt{s} - 2|\vec{p}_3|)\, \frac{1}{4\, s/2} |\mathcal{M}|^2$$

$\delta(a -)$

$$d\sigma = \int \frac{dp\, p^2}{(2\pi)^2 4p^2} \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi\, \delta(\sqrt{s} - 2p_3)\, \frac{1}{2s} |M|^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s} |M|^2$$

Including the sum over spins and the sum over colors, we have

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s} (1 + \cos^2\theta) . e^4 N_c \sum_q e_q^2$$

$$e^4 = 16\pi^2 \alpha^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi \alpha^2}{2s} (1 + \cos^2\theta) N_c \sum_q e_q^2$$

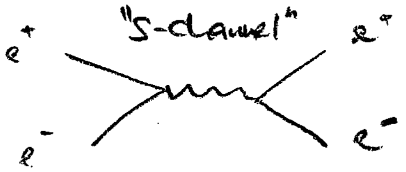
$$\boxed{\sigma^{tot} = \frac{\pi \alpha^2}{2s} \cdot \frac{8}{3} N_c \sum_q e_q^2}$$

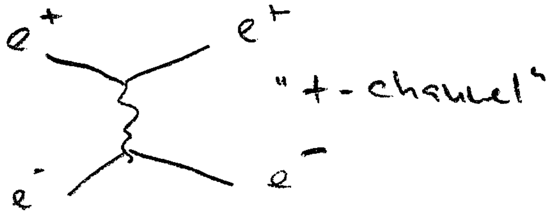
It is common to look at the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

$$= N_c \sum_q e_q^2 \cdot \{1 + O(\alpha_s)\}$$

Note that $e^+e^- \to e^+e^-$ has two contributions

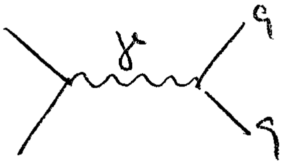




The t-channel diagram does not exist for $e^+e^- \to \mu^+\mu^-$

We have neglected contributions due to Z-exchange, which is fine at low energies

$|M|^2 = \Big|$



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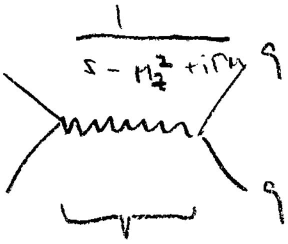

$\Big|^2$

see Ellis's book for the full expressions.

$\sim s/M_Z^2$ for $q^2 \to 0$

We have also neglected quark (and electron) masses. Including the mass in our calculation is straightforward, but makes both the phase-space integrals and the amplitude more complicated.

In terms of the quark velocity $\beta = \frac{v_{c.m.}}{c} = \sqrt{1-\frac{4m_Q^2}{s}}$, one finds

$$R_{e^+e^- \to \bar{Q}Q} = N_c \, e_Q^2 \, \beta \, \frac{3-\beta^2}{2}$$

$$= N_c \, e_Q^2 \left[1 - \frac{6 m_Q^4}{s^2} + O\left(\frac{m_Q^6}{s^3}\right) \right]$$

so the masses of light quarks can be safely neglected. (If R would depend strongly on these masses, this would spell trouble, i.e. non-perturbative physics.)

Note that heavy quarks with $2m_Q > \sqrt{s}$ cannot be produced, so the sum over flavors in the R-ratio should not contain those.

Looking at the data*, the agreement is quite nice, except at very low energy or near thresholds $\sqrt{s} \cong 2m_Q$, where Q is a charm or bottom quark.

In fact, the R-ratio provides a great illustration of asymptotic freedom: the hadronic cross section at high energies is well approximated by the cross section e^+e^- to a pair of free quarks.

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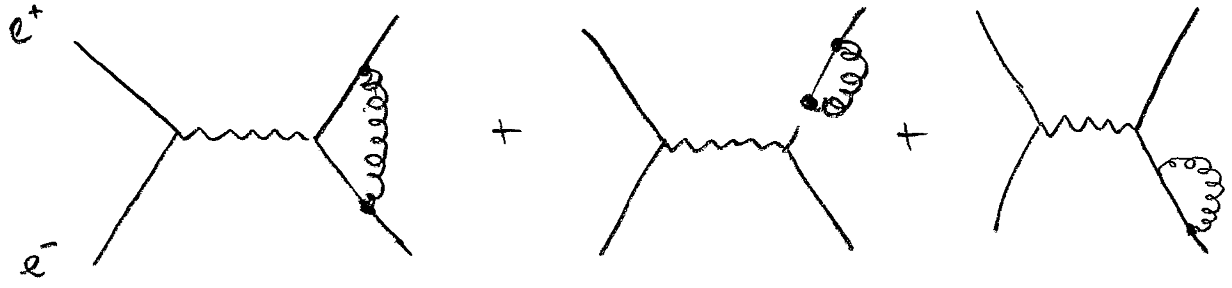
* See slides

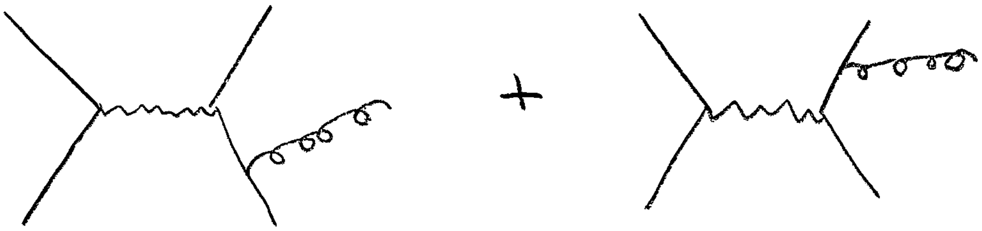
Let us now look at perturbative corrections.

At order α_s, there are two types of corrections:

1.) Virtual corrections: one-loop corrections to $e^+e^- \to q\bar{q}$



2.) Real emission diagrams



Their evaluation involves an interesting subtlety, which we'll discuss shortly, but let's first look at the result.

One finds that

$$R = N_c \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s}{\pi} \right\}$$

The result is known to $O(\alpha_s^4)$! At the next order, evaluating the diagrams gives

$$R = N_c \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 \right.$$

$$\left. \cdot \left[C_2 + \frac{\beta_0}{4} \left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln(s) \right) \right] \right\}$$

$$= \; !$$

with $C_2 = \left(\frac{2}{3} \zeta_3 - \frac{11}{12} \right) n_f + \frac{365}{24} - 11 \zeta_3$

$$= 1{,}99 - 0{,}115 \, n_f \, .$$

Here $\alpha_s \equiv \alpha_s^0$ the bare coupling.

Renormalize $\alpha_s^0 = \alpha_s(\mu) Z_g^2 \mu^{2\varepsilon}$

$$= \alpha_s(\mu) \left[1 - \frac{\alpha}{4\pi} \frac{\beta_0}{2\varepsilon} \right]^2 \mu^{2\varepsilon}$$

$$\mu = \mu_{\overline{MS}} \, e^{-\gamma_E/2} \, (4\pi)^{+1/2}$$

$$R = N_c \sum_q e_q^2 \left\{ 1 + \left(\frac{\alpha_s(\mu)}{\pi}\right) + \left(\frac{\alpha(\mu)}{\pi}\right)^2 \left[C_2 - \frac{\beta_0}{4} \ln \frac{s}{\mu^2} \right] \right\}$$

Note:

$$\frac{d}{d\ln\mu}\left(\frac{\alpha(\mu)}{\pi}\right) = \frac{1}{\pi}\left(-2\alpha_s(\mu)\beta_0 \frac{\alpha_s}{4\pi}\right) = \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[-\frac{\beta_0}{2} \right]$$

$$\Rightarrow \frac{d}{d\ln\mu} R = N_c \sum_q e_q^2 \left\{ \left(\frac{\alpha}{\pi}\right)^2 \left(-\frac{\beta_0}{2}\right) + \frac{\beta_0}{2}\left(\frac{\alpha}{\pi}\right)^2 \right\} + O(\alpha^3)$$

$$= O(\alpha^3)$$

So R is scale invariant up to the order we have calculated.

However, if we choose $\mu^2 \gg s$ or $\mu^2 \ll s$, the $\ln\left(\frac{s}{\mu^2}\right)$ term will become very large and perturbation theory breaks down, so we should choose $\mu^2 \approx s$. Often people vary $\mu \in \left[\frac{\sqrt{s}}{2}, 2\sqrt{s}\right]$ to estimate the uncertainty from higher orders.