3.5 The β-function and the running of α_s

The correlation functions of QCD contain UV divergences, which we are regulating by keeping $d \neq 4$. As we have seen in the example of the fermion self-energy, the divergences manifest themselves as $\frac{1}{\varepsilon}$ poles, where $d = 4 - 2\varepsilon$.

In order to eliminate the divergences into the parameters of the theory, one defines

$$A^a_{0\mu} = Z_3^{1/2} A^a_\mu \qquad q_0 = Z_q^{1/2} q$$

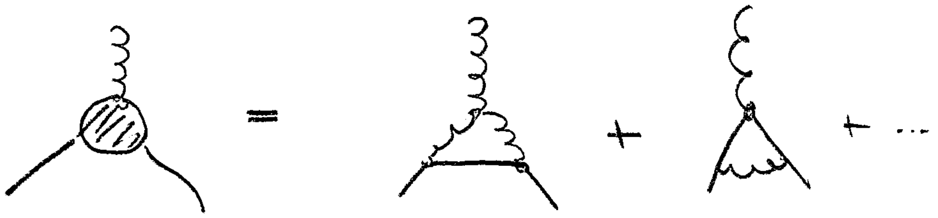
$$g_0 = Z_g \, g_s \, \mu^\varepsilon \qquad m_0 = Z_m m$$

bare ↗ ↖ renormalized.

(+ gauge parameter, ghost field)

The renormalization scale μ is arbitrary. The factor μ^ε guarantees that g_s is dimensionless.

By calculating



one detrmines

$$Z_g = 1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A - \frac{2}{6}\overset{\text{"number of quarks"}}{\overset{\swarrow}{n_f}}\right)\frac{1}{\varepsilon}$$

$$\alpha_s = g^2/4\pi$$

Note that we have decided <u>not</u> to include any finite higher order terms into the Z-factor.

This is called "minimal subtraction" (MS).

We would like to study the behavior of $g(\mu)$ when we change the scale μ.

Define $$\mu\frac{d}{d\mu}g(\mu) = \frac{d}{d\ln\mu}g(\mu) = \beta(g(\mu),\varepsilon)$$

Using that the bare coupling is μ - independent, we can obtain the β- function from the Z_g-factor

$$\frac{d}{d\ln\mu} g_0 = 0 = \frac{d}{d\ln\mu} Z_g\, g\, \mu^{\varepsilon}$$

so

$$\left(\frac{d}{d\ln\mu} Z\right) g \mu^{\varepsilon} + Z\, \beta(g,\varepsilon)\, \mu^{\varepsilon} + \varepsilon\, Z\, g\, \mu^{\varepsilon} = 0$$

$$\Longrightarrow \quad \beta(g,\varepsilon) = \underbrace{-g\,\frac{1}{Z}\,\frac{d}{d\ln\mu} Z}_{} - \varepsilon g$$

$$= \underbrace{\beta(g)}_{4\text{-d }\beta\text{-function}} - \varepsilon g$$

Note that $Z = 1 + \frac{1}{\varepsilon} Z_{g,1} + \frac{1}{\varepsilon^2} Z_{g,2} + \dots$

only depends implicitly on μ via $g(\mu)$:

$$\frac{d}{d\ln\mu} Z = \frac{dZ}{dg}\,\frac{dg}{d\ln\mu}$$

In the MS scheme in dimensional regularization the β-function can be obtained from the $\frac{1}{\varepsilon}$ part of the Z-factor:

$$\boxed{\beta(g) = 2g^3 \frac{dZ_{g,1}}{dg^2}} \qquad Z_g = 1 + \frac{1}{\varepsilon} Z_{g,1} + \frac{1}{\varepsilon^2} Z_{g,2} + \dots$$

To show this use

$$Z_g \, \beta(g) = -g \frac{dZ}{d\ln\mu} = -g \frac{dZ}{dg} \frac{dg}{d\ln\mu}$$

$$= -g \frac{dZ}{dg} \beta(g, \varepsilon)$$

$$= -g \frac{dZ}{dg} \left(\beta(g) - \varepsilon g \right)$$

Now take the $O(\varepsilon^0)$ coefficient of this equation

$$\beta(g) = +g^2 \frac{dZ_{g,1}}{dg} = 2g^3 \frac{dZ_{g,1}}{dg^2}$$

Using $Z_g = 1 - \frac{g^2}{16\pi^2}\left(\frac{11}{6}C_A - \frac{2}{6}n_f\right)\cdot\frac{1}{\varepsilon}$

we have

$$\beta(g) = -g\frac{g^2}{16\pi^2}\left(\frac{11}{3}C_A - \frac{2}{3}n_f\right) + O(g^5)$$

$$= -g\frac{\alpha_s}{4\pi}\beta_0$$

In terms of α_s

Gross & Wilczek '73 Politzer '73
Nobel price '99

$$\frac{d\alpha_s}{d\ln\mu} = -2\alpha_s\left[\beta_0\left(\frac{\alpha_s}{4\pi}\right) + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \beta_2\left(\frac{\alpha_s}{4\pi}\right)^3 + \beta_3\left(\frac{\alpha}{4\pi}\right)^3\right]$$

Ritbergen, Larin, Vermaseren '97

Note $\beta_0 = 11 - \frac{2}{3}n_f > 0$ for $n_f < 17$
$= 7$ in QCD with 6 quarks

In QED $\beta_0 = -\frac{2}{3}n_f < 0$

It turns out that only non-abelian gauge theories can have $\beta_0 < 0$. In fact, Gross & Wilczek wanted to prove that all QFTs have $\beta < 0$.
Fortunately they failed!

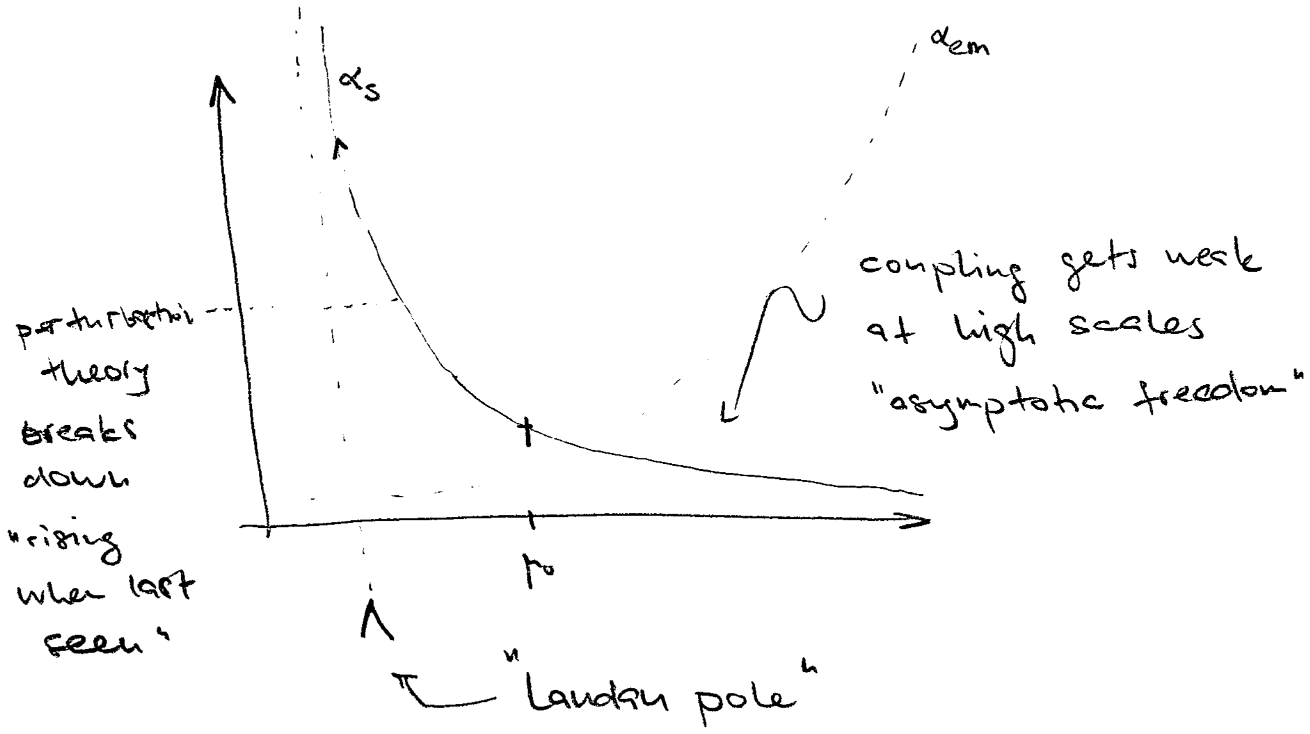
To understand the meaning of $\beta_0 > 0$, let's solve the RG equation for α_s

$$\frac{d}{d\ln\mu}\alpha_s = -2\alpha_s\beta_0\frac{\alpha_s}{4\pi} = \beta(\alpha_s)$$

$$\Rightarrow \int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)} = \int_{\mu_0}^{\mu} d\ln\mu = \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)} = -\frac{4\pi}{2\beta_0}\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{\alpha^2} = +\frac{4\pi}{2\beta_0}\left(\frac{1}{\alpha(\mu)} - \frac{1}{\alpha(\mu_0)}\right)$$

$$\rightsquigarrow \alpha(\mu) = \frac{\alpha(\mu_0)}{1+\frac{\alpha(\mu_0)}{4\pi}\beta_0\ln\left(\frac{\mu^2}{\mu_0^2}\right)} \equiv \frac{4\pi}{\beta_0\ln\left(\frac{\mu^2}{\Lambda^2}\right)}$$



Note that the coupling not only depends on the renormalization scale μ, but also on the renormalization scheme.

The standard scheme is the $\overline{MS}$ scheme, in which some universal finite $\ln(4\pi) + \gamma_E$ factors are absorbed into the Z-factors.

$$\mu_{\overline{MS}} = \mu_{MS} \cdot e^{\gamma_E/2} (4\pi)^{-1/2}$$

$$\alpha_{MS}(\mu) = \alpha_{\overline{MS}}(\mu) \left(1 + \beta_0 (\gamma_E - \ln 4\pi) \frac{\alpha_{\overline{MS}}(\mu)}{4\pi}\right) + \ldots$$

For a review of the determination of α_s, see S. Bethke 0908.1135.