3.4 From Green's functions to cross sections

We have derived the Feynman rules for vacuum expectation values of time-ordered products of fields. From these, one can extract the scattering amplitudes.

The propagation of a particle manifests itself as a pole in the Fourier transformed correlation function, e.g.

on-shell wave function renorm.

$$\int d^4x \, \langle \phi(x)\, \phi(0) \rangle = \frac{iZ}{p^2 - m_{phys} + i\varepsilon} + \dots$$

To show this, use

$$\langle \phi(x)\, \phi(0) \rangle = \langle 0 | \phi(x)\, \phi(0) | 0 \rangle \, \Theta(x^0) + \langle 0 | \phi(0)\, \phi(x) | 0 \rangle \, \Theta(-x^0)$$

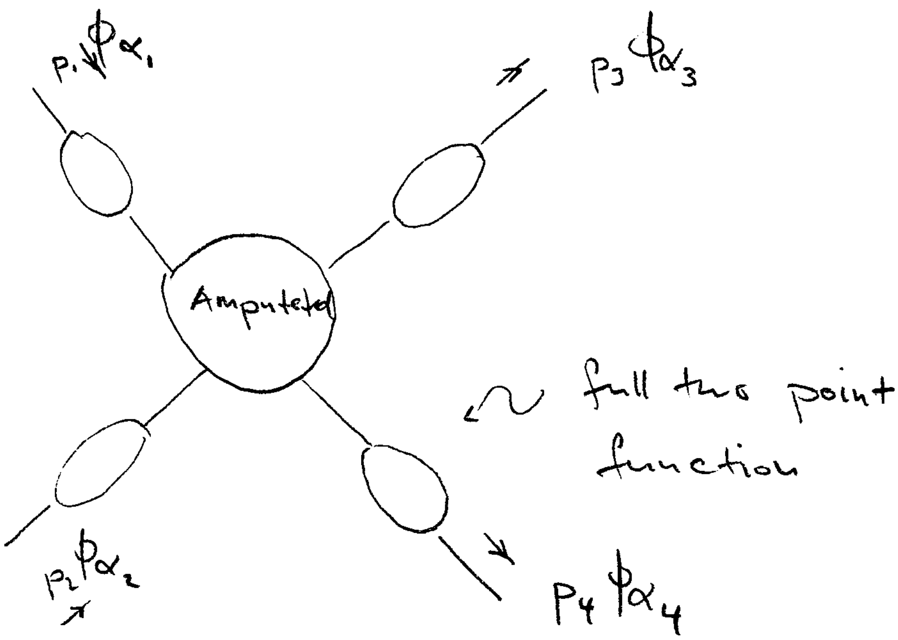
and insert a complete set of states

$$\sum_X |X\rangle\langle X| = |0\rangle\langle 0| + \int \frac{d^3p}{(2\pi)^3 2E} |p\rangle\langle p| + \dots$$

one finds $\sqrt{Z} = \langle 0|\phi(0)|p\rangle$.

For a free theory $m_{phys} = m$, $Z = 1$.

n-point correlation functions have the form



The indices $\alpha_1, \dots, \alpha_4$ are the Dirac or Lorentz indices of the fields $\phi_{\alpha_i} = A_{\alpha_i}, \psi_{\alpha_i}$ or $\bar{\psi}_{\alpha_i}$

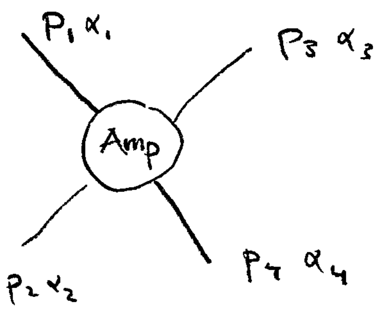
The scattering amplitude, or S-matrix, is given by

$$\langle p_3, s_3 ; p_4, s_4 \,|\, S \,|\, p_1, s_1 ; p_2, s_2 \rangle =$$

$$\sum_{\{\alpha\}} \langle p_3, s_3 | \phi_{\alpha_3}^{(0)} | 0 \rangle \langle p_4, s_4 | \phi_{\alpha_4}^{(0)} | 0 \rangle$$

$$\langle 0 | \phi_{\alpha_1}(0) | p_1, s_1 \rangle \langle 0 | \phi_{\alpha_2}(0) | p_2, s_2 \rangle$$

$*$



We have

These can be determined by calculating the full propagator

$$\langle 0 | \psi_\alpha(0) | p, s \rangle = \sqrt{Z_\psi}\, u_\alpha(p,s)$$

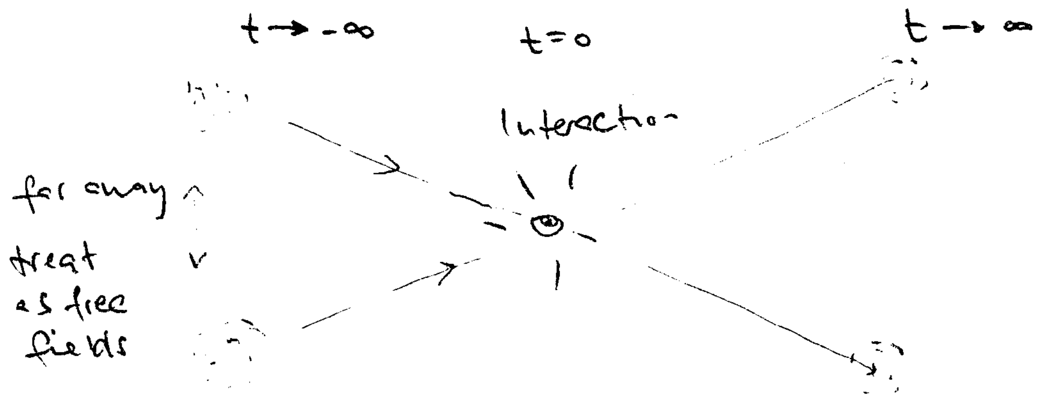
$$\langle 0 | \bar{\psi}_\alpha(x) | p, s \rangle = \sqrt{Z_\psi}\, \bar{v}_\alpha(p,s)$$

$$\langle 0 | A^a_\mu | p, \lambda, b \rangle = \sqrt{Z_A}\, \varepsilon_\mu(p,\lambda)\, \delta^{ab}$$

[I have not written out the color quantum numbers of the quarks, it would look like in the gluon case]

This result is called the LSZ reduction formula. It's derivation is quite subtle.

To derive it one takes the Fourier transform of the correlation function, but to keep the different particles separated one has to work with wave packets



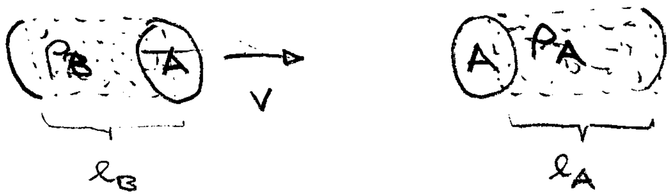
For theories with massless fields, the assumptions going into the derivation cannot really be fulfilled, e.g. an electron will always emit soft photons, so we cannot really prepare a wave packet with just an electron inside. In QCD things are even worse: how do you prepare a wave packet with a single quark inside?

There is a punishment for using the LSZ formula naively for QED and QCD: the scattering amplitudes contain infrared singularities.

In suitably defined observables, these IR singularities will be absent. Finding such "suitably" defined observables is what will keep us ~~busy~~ busy for much of the rest of the lecture.

Now that we have the scattering amplitude, we can calculate scattering cross sections.

Take two bunches of particles at a collider



define $$\sigma = \frac{\text{"number of events"}}{\rho_A \, l_A \, \rho_B \, l_B \, A}$$

where A is the overlap area.

$$[\sigma] = \frac{1}{\frac{1}{m^3} \, m \, \frac{1}{m^3} \, m \, m^2} = m^2$$

The cross section has units of an area.

Assuming that both beams have area A and overlap completely, we can also write

$$N = \text{"Number of events"} = \frac{N_A N_B}{A} \cdot \sigma$$

Instead of cm^2, cross sections are usually given in barns: $1b = 1$ "barn" $= 10^{-24}$ m^2

However cross sections measured at present colliders are typically in the $fb = 10^{-15}$ barn range.

An important quantity is the luminosity $\mathcal{L}$

$$\frac{dN}{dt} = \mathcal{L}\sigma .$$

In our example $\frac{N_A \cdot N_B}{A} = \int dt\, \mathcal{L}$

is called the integrated luminosity. It is measured in inverse femtobarn. $(fb)^{-1}$.

So with $7 (fb)^{-1}$ integrated luminosity, you'll get 7000 events for a 1pb cross section.

Run II at the Tevatron has collected around $7\ fb^{-1}$ as of now.

See slides for some measured cross sections.

To calculate the cross section one splits the S-matrix into $S = \mathbb{1} + iT$.

The $\mathbb{1}$ operator is relevant for the part of the S-matrix in which no scattering happens.

Then one defines

$$\langle p_1, \dots, p_n | iT | k_1, k_2 \rangle = (2\pi)^4 \delta(k_1 + k_2 - \sum_{i=1}^{n} p_i)$$
$$i\mathcal{M}(k_1, k_2 \to p_1, \dots, p_n)$$

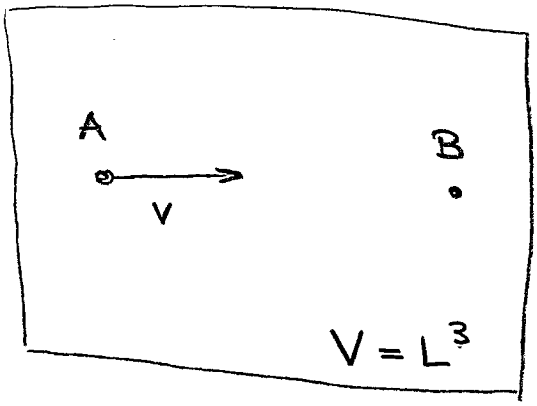
The differential cross section is given by

$$d\sigma = \frac{1}{4\sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2}} \left(\prod_{i=1}^{n} \frac{d^3 p_i}{2E_i (2\pi)^3} \right) (2\pi)^4 \delta(k_1 + k_2 - \Sigma p_i)$$
$$|\mathcal{M}(k_1, k_2 \to p_1, \dots p_n)|^2$$

For a careful derivation of this formula, one needs to consider wave packets.

Let's avoid this by considering scattering in a box of volume V during a time period T.

Let's assume that the box contains only two particles:



Either there will be scattering or not. The probability for scattering to some final state is

$$P = \frac{|\langle p_1, \dots, p_n | iT | k_A, k_B \rangle|^2}{\langle k_A | k_B \rangle \langle k_2 | k_2 \rangle \prod_{i=1}^{n} \langle p_i | p_i \rangle}$$

Here $P \equiv$ "number of events"

We had defined

$$\sigma = \frac{\text{" number of events "}}{\rho_A \, l_A \, \rho_B \cdot l_B \, A}$$

$$= \frac{P}{\frac{1}{V} L \frac{1}{V} \cdot V} = \frac{P}{\frac{L}{V}} = \frac{P}{\underbrace{\frac{|\vec{v}_A| T}{V}}_{\text{flux}}}$$

Let's now evaluate P

$$P = \frac{(2\pi)^4 \delta(k_A + k_B - \sum_i p_i)\,(2\pi)^4 \delta^4(0)\, |\mathcal{M}|^2}{2E_A (2\pi)^3 \delta^3(0)\; 2E_B (2\pi)^3 \delta^3(0) \prod_{i=1}^{n} (2\pi)^3\, 2E_i\, \delta^3(0)}$$

$$\left[(2\pi)^3 \delta^3(0) = \int d^3x\, e^{i 0 \cdot x} = V \qquad (2\pi)^4 \delta^4(0) = V \cdot T \right.$$

$$= \frac{(2\pi)^4 \delta(k_A + k_B - \sum_i p_i)\, |\mathcal{M}|^2 \cdot V \cdot T}{4 E_A E_B\, V^2 \prod_{i=1}^{n} 2E_i\, V}$$

This is the probability for scattering into a state with definite momentum.

Let's sum over momenta in some range.

In our box, the momenta are quantized

$\vec{k}_i = \frac{2\pi}{L} \vec{n}_i$, where $\vec{n}_i$ is a vector of

integers. For a large volume $V = L^3$

$$\sum_{n_i} = \frac{V}{(2\pi)^3} \int d^3 k_i$$

$$dP = \frac{(2\pi)^4 \delta(k_A + k_B - \sum_i p_i) |\mathcal{M}|^2}{4 E_A E_B \cdot V / T} \cdot \prod_{i=1}^{n} \frac{d^3 k_i}{2E_i (2\pi)^3}$$

$$d\sigma = \frac{dP \cdot V}{|\vec{v}_A| \cdot T} \qquad \left[E_B = m_B \qquad v = \frac{|\vec{p}_A|}{E_A} \right]$$

$$= \frac{(2\pi)^4 \delta(k_A + k_B - \sum_i p_i) |\mathcal{M}|^2}{4 m_B \cdot |\vec{p}_A^{\,lab}|} \prod_{i=1}^{n} \frac{d^3 k_i}{2E_i (2\pi)^3}$$

Note $|\vec{p}_A| m_B = \sqrt{(k_A \cdot k_B)^2 - m_A^2 m_B^2}$

With the expression on the RHS, we recover the cross section in an arbitrary frame.