3.3. Dimensional Regularization

The Feynman diagrams which appear in the perturbative series contain UV divergent loop integrals. To make sense of these integrals, one needs to regularize the theory.

If the divergencies can be absorbed into a redefinition of the parameters of the theory (couplings, masses, gauge parameters, ...), physical predictions will be meaningful, despite the presence of these divergences. This process is called renormalization.

It is desirable that the regularization preserves the symmetries of the theory. If not, one has to carefully recover the symmetry in the renormalization process, which can be very difficult.

It is especially important, that gauge symmetry is respected by the regularization. To my knowledge only to regularizations achieve this

1.) lattice regularization à la Wilson

2.) dimensional regularization
('t Hooft and Veltman '72)

Lattice regularization breaks Lorentz invariance, but leads to a nonperturbative definition of the theory.

Dim. reg. respects all symmetries, except chiral symmetry (problems with γ_5 & $\varepsilon_{\mu\nu\rho\sigma}$).

The idea of dim. reg. is simple to state: evaluate all loop integrals in d dimensions. For small enough d, the integrals will converge.

Let us do a sample calculation to illustrate the technique.

$$k^0 \to i k_E^0$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{(k^2)}{(m^2 - k^2 - i\varepsilon)^n} = i \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(m^2 + k^2)^n}$$

$$= i \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty dk_E \, k_E^{d-1} \frac{1}{(k_E^2 + m^2)^n}$$

The area of a d-dim sphere is

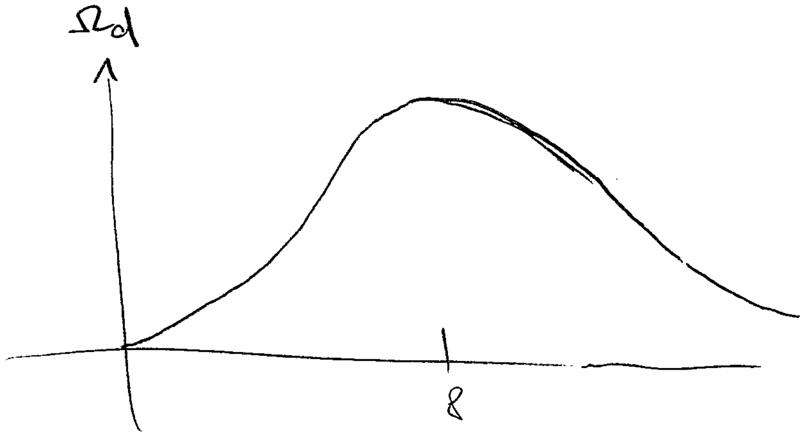
$$(\sqrt{\pi})^d = \left(\int dx \, e^{-x^2} \right)^d = \int d^d x \exp\left(-\sum_{i=1}^{d} x_i^2 \right)$$

$$= \int d\Omega_d \int_0^\infty dx \, x^{d-1} e^{-x^2} = \int d\Omega_d \int_0^\infty dx^2 \frac{1}{2} (x^2)^{\frac{d}{2}-1} e^{-x^2}$$

$$= \left(\int d\Omega_d \right) \frac{1}{2} \Gamma(d/2)$$

$$\Longrightarrow \quad \Omega_{d/2} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

d	Ω_d
1	2
2	2π
3	4π
4	$2\pi^2$
∞	0



Also the k-integral gives a Γ-function

$$\int_0^\infty dk\, k^{d-1} \frac{(k^2)^\alpha}{(k^2+m^2)^\beta} = (m^2)^{\alpha-\beta+\frac{d}{2}} \frac{\Gamma(\frac{d}{2}+\alpha)\Gamma(\beta-\frac{d}{2}-\alpha)}{2\Gamma(\beta)} \quad (*)$$

$$\int_0^\infty d\hat{k}\, k^\gamma (k^2+1)^\beta = \frac{1}{2}\int_0^1 dx\, (1-x)^{\overset{B-1}{\frac{\gamma-1}{2}}} x^{\overset{A-1}{\beta-2+\frac{1-\gamma}{2}}}$$

$$x = \frac{1}{k^2+1} \quad ; \quad k^2 = \frac{1-x}{x} \Bigg] = \frac{1}{2} B\left(\frac{\gamma+1}{2}, \beta-\frac{\gamma+1}{2}\right)$$

$$\int_0^1 dx\, x^\alpha (1-x)^\beta = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Note that the integral on the LHS of $(*)$ is only defined if

$$d + 2\alpha < 2\beta$$

$$d + 2\alpha > 0$$

However, the expression on the RHS is well defined for arbitrary complex d, except for poles at $\frac{d}{2}+\alpha = 0, -1, -2, \ldots$ and

$$\beta-\alpha-\frac{d}{2} = 0, -1, -2, \ldots$$

Another interesting property is that the RHS vanishes for $\beta \to 0$. We now <u>define</u> the integral by the RHS:

$$(**) \quad \int d^d k \, \frac{(k^2)^\alpha}{(m^2 - k^2 - i\varepsilon)^\beta} := i \pi^{d/2} (m^2)^{d/2+\alpha-\beta} \, \frac{\Gamma(\alpha + \frac{d}{2}) \Gamma(\beta - \alpha - \frac{d}{2})}{\Gamma(d/2) \Gamma(\beta)}$$

This implies, e.g.

$$\int d^d k \, (k^2)^\alpha = 0$$

For one-loop integrals involving several propagators, we can use the Feynman parameterization to combine them to the form

$$\int d^d k \, \frac{\{1, k_\mu, k_\mu k_\nu, \dots\}}{(M^2 - k^2 - i\varepsilon)^n} \qquad g^\mu{}_\mu = d$$

$$= \int d^d k \, \frac{\{1, 0, k^2 \frac{g^{\mu\nu}}{d}, \dots\}}{(M^2 - k^2 - i\varepsilon)^n}$$

where M depends on the Feynman parameters, external momenta, and masses.

Feynman parameterizations:

$$\int_0^1 dx \frac{1}{[Ax + B(1-x)]^2} = \frac{1}{AB}$$

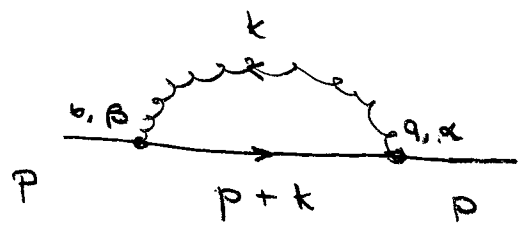
$$\int_0^1 dx_1 \dots dx_n \, \delta(\Sigma x_i - 1) \frac{\prod x_i^{m-1}}{[\Sigma x_i A_i]^{\Sigma m_i}} \cdot \frac{\Gamma(m_1 + \dots + m_n)}{\Gamma(m_1) \dots \Gamma(m_n)}$$

$$= \frac{1}{A_1^{m_1} A_2^{m_2} \dots A_n^{m_n}}$$

In collider physics calculations, we encounter not only UV, but also infrared divergences, as we'll see soon. Also in these cases dim. reg. is used as a regulator.

A sample diagram

Consider the quark self-energy



The symmetry factor is 1 ; 2 contractions and $\frac{1}{2!}$ because we have two identical vertices. Let's calculate the amputated diagram, i.e. the diagram without external propagators.

$$-i\Sigma = \int \frac{d^dk}{(2\pi)^d} (ig\gamma^\alpha t^a) \cdot \frac{i}{\not p + \not k - m_q + i\varepsilon} (ig\gamma^\beta t^b)$$

$$\cdot \frac{i}{k^2 + i\varepsilon}\left[- g_{\alpha\beta} + \frac{k_\alpha k_\beta}{k^2 + i\varepsilon}(1-\xi)\right] \delta^{ab}$$

Color structure: $t^a t^b \delta^{ab} = t^a t^a = C_F \mathbb{1}$

For simplicity let's use Feynman gauge $\xi = 1$.

$$-i\Sigma = -g^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2[(p+k)^2 - m^2]}$$

$$\cdot \gamma^\alpha [\not{p} + \not{k} + m] \gamma_\alpha$$

Dirac structure:

$$\gamma_\alpha \gamma^\alpha = \frac{1}{2}\{\gamma_\alpha, \gamma^\alpha\} = g_\alpha{}^\alpha = d$$

$$\gamma_\alpha \gamma^\mu \gamma^\alpha = \gamma_\alpha \{\gamma^\mu, \gamma^\alpha\} - \gamma_\alpha \gamma^\alpha \gamma^\mu$$

$$= \gamma_\alpha 2g^{\mu\alpha} - d\gamma^\mu = (2-d)\gamma^\mu$$

$$-i\Sigma = -g^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{(2-d)(\not{p}+\not{k}) + d\cdot m}{k^2[(p+k)^2 - m^2]}$$

This expression contains two integrals

$$\{I, I_\mu\} = \int d^dk \frac{\{1, (p+k)^\mu\}}{k^2[(p+k)^2 - m^2]}$$

$$\{I, I_\mu\} = \int_0^1 dx \int d^dk \; \frac{\{1, (p+k)^\mu\}}{[\underbrace{k^2 + 2xp.x}_{(k+xp)^2 - x^2p^2} + x p^2 - x m^2]}$$

Shift $k \to k - xp$

$$\{I, I_\mu\} = \int_0^1 dx \int d^dk \; \frac{\{1, \; p^\mu - x p^\mu + \underbrace{\cancel{k^\mu}}_{0 \text{ (odd)}}\}}{[k^2 - M^2]^2}$$

where $M^2 = + x m^2 - x(1-x) p^2$

In this form, we can use our previous result for the k-integration.

$$\{I, I_\mu\} = \int_0^1 dx \{1, (1-x) p^\mu\} \; i\pi^{d/2} \, \Gamma(2 - d/2)$$
$$\cdot \left(x m^2 - x(1-x) p^2 \right)^{d/2 - 2}$$

$d = 4 - 2\varepsilon$

$$= i\pi^{d/2} \left(\frac{1}{\varepsilon} - \gamma_E\right) \int_0^1 dx \{1, (1-x) p^\mu\}$$
$$\times \left(x m^2 - x(1-x) p^2 \right)^{-\varepsilon}$$

$$= i\pi^{d/2} \frac{1}{\varepsilon} \left\{ 1, \frac{p^\mu}{2} \right\} + \ldots$$

We can plug this back in to get the divergent part of the quark self-energy in Feynman gauge $\xi = 1$.

$$-i\Sigma(p,m) = -i\left(\frac{\alpha_s}{4\pi}\right)C_F\left[-\frac{1}{\varepsilon}\not{p} + \frac{4}{\varepsilon}m\right] + \dots$$

$\alpha_s = \frac{g^2}{4\pi}$, in analogy to $\alpha = \frac{e^2}{4\pi}$

For $\alpha_s \to \alpha$, $C_F \to 1$, we recover the QED result.