## 3.2. Feynman rules

Let's briefly recapitulate how they are derived for a scalar theory (for the purpose of Feynman rules, we can view each component of the gluon field as a scalar field).

E.g. $$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}}_{\mathcal{L}_0} \underbrace{- \frac{\lambda}{4!}\phi^4}_{+\ \mathcal{L}_I}$$

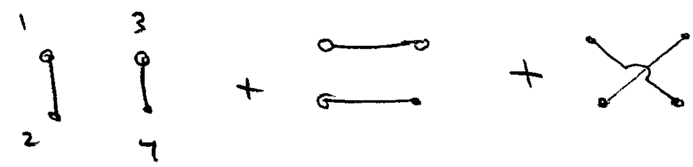
Define

$$\langle \phi_1 \dots \phi_n \rangle = \langle \phi(x_1) \cdots \phi(x_n) \rangle = \int \mathcal{D}\phi\, \phi(x_1) \dots \phi(x_n)\, e^{i\int d^4x\, \mathcal{L}_0}$$

Wick's theorem

$$\langle \phi_1 \cdots \phi_n \rangle = \sum_{\substack{\text{all pairings} \\ i_1 \dots i_n}} \underbrace{\langle \phi_{i_1} \phi_{i_2} \rangle}_{\text{free propagator.}} \langle \phi_{i_3} \phi_{i_4} \rangle \cdots \langle \phi_{i_{n-1}} \phi_{i_n} \rangle$$

For example

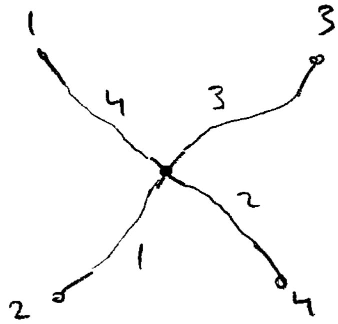
$\langle \phi_1 \cdots \phi_4 \rangle =$ 

Interacting theory

$$\langle \phi_1 \cdots\cdots \phi_4 \, e^{i\int d^4x \, \mathcal{L}_I} \rangle = \langle \phi_1 \cdots \phi_4 \rangle - i \int d^4x \frac{\lambda}{4!} \langle \phi_1 \cdots \phi_4 \, \phi_x^4 \rangle$$

$$- \frac{1}{2!} \int d^4x \int d^4y \left(\frac{\lambda}{4!}\right)^2 \langle \phi_1 \cdots\cdots \phi_4 \, \phi_x^4 \, \phi_y^4 \rangle$$

At $O(\lambda)$, we have




4! contractions

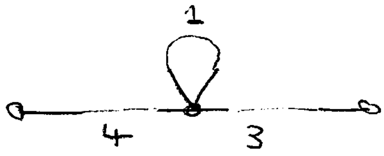
Two conventions for Feynman rules

A.) The Feynman rule is $-i\frac{\lambda}{4!}$ and we have to count the number of contractions

B.) The Feynman rule is $-i\lambda$. Sometimes we have to divide by a symmetry factor because not all contractions arise.

As far as I know only (B.) is used in the QCD literature.

Example : 

A.) 


$$-i\frac{\lambda}{4!}\,4\cdot 3 = -i\frac{\lambda}{2}$$

B.) 

$$-i\lambda \cdot \frac{1}{2}$$ ← symm. factor

Most of the time, diagrams are calculated in momentum space. To obtain the Feynman rules one Fourier transforms the Lagrangian

incoming momenta

$$-\frac{\lambda}{4!}\int d^4x\,\phi^4(x) = -\frac{\lambda}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\int d^4x\, e^{-i(k_1 \dots k_4)x}\,\tilde{\phi}(k_1)\dots\tilde{\phi}(k_4)$$

$$\left[\int_k = \int\frac{d^4k}{(2\pi)^4}\right]$$

$$= -\frac{\lambda}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}(2\pi)^4\,\delta(k_1+k_2+k_3+k_4)\,\tilde{\phi}(k_1)\dots\tilde{\phi}(k_1)$$

At tree-level all momentum integrations can be performed trivially, eliminating $\delta$-functions.

At higher orders nontrivial integrations over loop momenta remain.

To obtain the QCD Feynman rules, we have to Fourier transform the Lagrangian.

Let's first split it into $\mathcal{L}_0 + \mathcal{L}_{Int}$

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \sum_q \bar{\psi}_q (i\not{\partial} - m_q)\psi_q - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$$

$$+ \bar{c}(-\partial^\mu D^{ab}_\mu)c$$

$$iD_\mu = i\partial_\mu + g A^a_\mu t^a$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g A^b_\mu A^c_\nu f^{abc}$$

$$D^{ab}_\mu = \partial_\mu \delta^{ab} + g f^{abc} A^c_\mu$$

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$$

$$+ \sum_q \bar{\psi}_q (i\not{\partial} - m_q)\psi_q + \bar{c}(-\partial^2)c$$

The free propagators for these fields are obtained by Fourier transforming and inverting. Remember that $i\partial_\mu \hat{=} k_\mu$.

$$\langle \Psi^a_{i\alpha}(x)\, \bar\Psi^a_{j\beta}(y)\rangle = \int_k \left(\frac{i}{\not k - m_q + i\varepsilon}\right)_{\alpha\beta} \delta_{ij}\, e^{-ik(x-y)}$$

$$\langle A^a_\mu(x)\, A^b_\nu(y)\rangle = \int_k \frac{i}{k^2+i\varepsilon}\left(-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\, e^{-ik(x-y)}$$

... same as photon prop.

$$\langle c^a(x)\, \bar c^b(y)\rangle = \int_k \frac{i}{k^2}\, \delta^{ab}\, e^{-ik(x-y)}$$

All propagators are color diagonal.

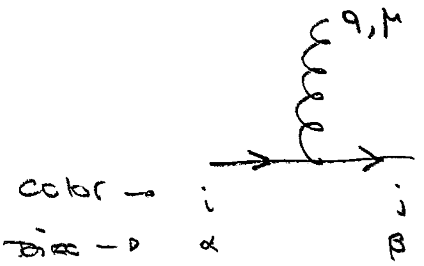
Now let's look at the interactions

$$\mathcal{L}_{Int} = g A^a_\mu \bar\Psi \gamma^\mu t^a \Psi - g f^{abc}(\partial^\beta A^a_\alpha) A^b_\beta A^c_\alpha$$

$$- \frac{g^2}{4} f^{abe} A^a_\alpha A^b_\beta f^{cde} A^c_\beta A^d_\alpha$$

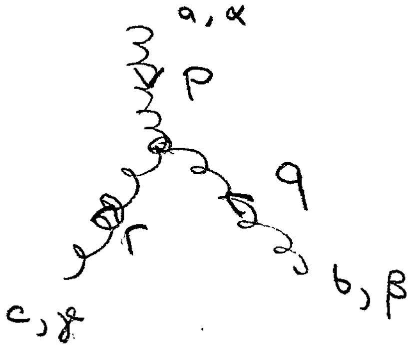
$$- \bar c^a g f^{abc} \partial^\mu A^c_\mu c^b$$

The first term gives the quark-gluon vertex



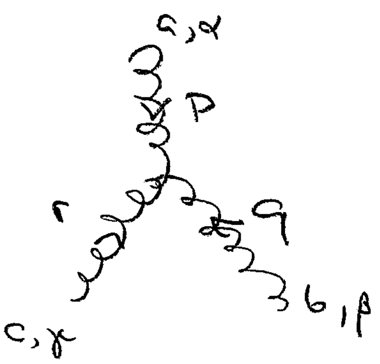

$$ig\,(\gamma^\mu)_{\beta\alpha}\,(t^a)_{ji}$$

The second is an interaction among three gluons, its Fourier transform is




$$-ig\,\frac{p^\beta}{i}\,g_{\alpha\gamma}$$

we follow convention B.), so the Feynman rule is the sum over 3! contractions, which correspond to the 3! permutations of the three gluon fields



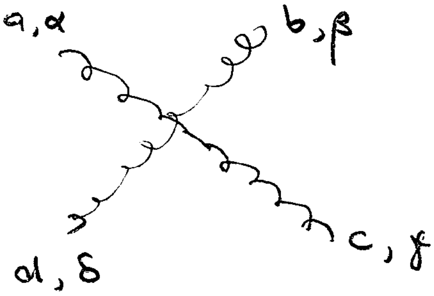

$$g\,f^{abc}\Big[(p-q)^\gamma g^{\alpha\beta}+(q-r)^\alpha g^{\beta\gamma}+(r-p)^\beta g^{\gamma\alpha}\Big]$$

this is the above term.

Now, the four gluon term

Fourier transform is

$$-\frac{g^2}{4} f^{abe} f^{cde} g_{\alpha\gamma} g_{\beta\delta}$$




In this case, there are 4! permutations, of which always 4 are equivalent.

- Feynman rule

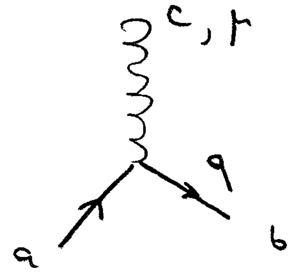
$$-ig^2 f^{abe} f^{cde} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})$$

$$-ig^2 f^{ace} f^{bde} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\beta\gamma})$$

$$-ig^2 f^{ade} f^{bce} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\gamma} g_{\beta\delta})$$

- Finally, for the gluon-ghost coupling, we find

$$-ig f^{abc} \frac{q^\mu}{i}$$

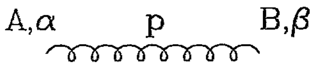



The complete set of Feynman rules are given on the next page.

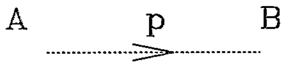
3.18.

[From K. Ellis, but with $g \to -g$]

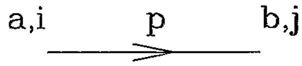
out $\xi$



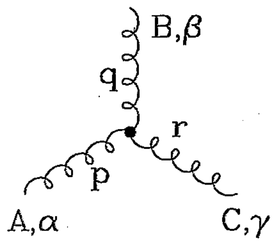

$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^\alpha p^\beta}{p^2+i\epsilon}\right]\frac{i}{p^2+i\epsilon}$$




$$\delta^{AB}\,\frac{i}{(p^2+i\epsilon)}$$

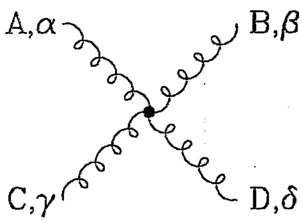



$$\delta^{ab}\,\frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

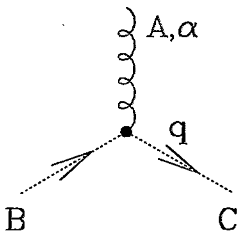



$$+g\,f^{ABC}[(p-q)^\gamma g^{\alpha\beta}+(q-r)^\alpha g^{\beta\gamma}+(r-p)^\beta g^{\gamma\alpha}]$$

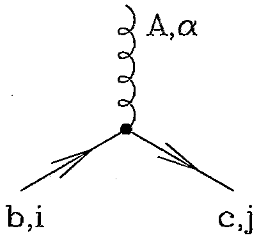
(all momenta incoming, $p+q+r = 0$)




$$\begin{aligned}&-ig^2\,f^{XAC}f^{XBD}\,[g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]\\&-ig^2\,f^{XAD}f^{XBC}\,[g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]\\&-ig^2\,f^{XAB}f^{XCD}\,[g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]\end{aligned}$$




$$-g\,f^{ABC}q^\alpha$$




$$+ig\,(t^A)_{cb}\,(\gamma^\alpha)_{ji}$$