3. Perturbative QCD

Now that we have constructed the Lagrangian of non-abelian gauge theory, we want to quantize these theories. By far the simplest method is to use the path integral formulation of QFT.

As in the case of QED the naive path integral over the gauge field

$$Z = \int \mathcal{D}A_\mu \exp\left[iS[A_\mu] \right]$$

is ill-defined. The problem arises from gauge invariance: all gauge configuration related by a gauge transformation have the same weight in the integral. In particular, all configurations where $A_\mu(x) = \frac{i}{g} V(x) \partial_\mu V(x)$; $V(x) = \exp(i\alpha^a t^a)$

have $S[A_\mu]=0$, since they are obtained from $A_\mu = 0$ by a gauge transformation. In order to get a meaningful path integral, we want to factor out the integral over the gauge group, since it will not contribute to gauge invariant expectation values.

A method to achieve this was found by Faddeev and Popov in '67. We'll first illustrate it for an ordinary integral before applying it to the path integral over the gauge field.

3.1. The Fadeev-Popov Lagrangian

Let's first consider a two-dimensional integral to illustrate the procedure:

$$I = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; f(x,y)$$

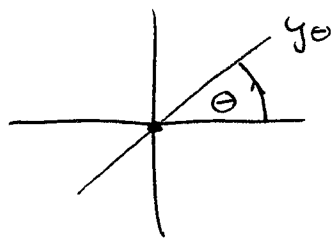
where $f(x,y)$ is rotation invariant ($\equiv$ gauge invariant in the QCD case). We want to write this as

$$I = \underbrace{\int_0^{2\pi} d\theta}_{\substack{\text{integral} \\ \text{over the symmetry group ("gauge group")}}} \int_0^{\infty} dr \; F(r)$$

How can we achieve this if we don't know $F(r)$?

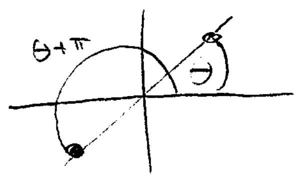
Fix a direction ($\equiv$ gauge) by the condition

$$y_\theta = x \sin\theta + y \cos\theta = 0$$



To be rotation invariant: integrate over all directions:

$$\int_0^{2\pi} d\theta \; \delta(y_\theta) \left| \frac{\partial y_\theta}{\partial \theta} \right| = 2$$



$$\frac{\partial y_\theta}{\partial \theta} = x \cos\theta - y \sin\theta = \sqrt{x^2+y^2}$$

$$\tan\theta = -\frac{y}{x}$$

Now insert this into the integral

$$I = \int d\theta \int dx \int dy \; \delta(y_\theta) \, \frac{1}{2} \sqrt{x^2+y^2} \, f(x,y)$$

and rotate your coordinate system by θ:

$$y' = y_\theta$$
$$x' = x\cos\theta - y\sin\theta$$

$$I = \int_0^{2\pi} d\theta \int dx' \int dy' \; \delta(y') \underbrace{\frac{1}{2}\sqrt{x'^2+y'^2} \, f(x',y')}_{\text{by rotation invariance}}$$

$$= \underbrace{2\pi}_{\substack{\text{volume} \\ \text{of the symmetry group}}} \times \int dx' \int dy' \; \delta(y') \, \frac{1}{2} \sqrt{x'^2+y'^2} \, f(x',y')$$

$$= 2\pi \int dx' \, \frac{x'}{2} \, f(x',0)$$

Now the real thing: consider the path integral

$$Z = \int \mathcal{D}A_\mu \exp\left[i \int d^4x \left(-\frac{1}{4} (F^i_{\mu\nu})^2 \right) \right]$$

(the complication only concerns the gauge part, so we don't include fermions for the moment.)

We now want to introduce the analogue of the ray $y_0 = 0$. First choose a gauge fixing condition

$$g(A) = \partial_\mu A^a_\mu = 0 \qquad (\text{"Lorentz gauge"})$$

or

$$g(A) = n_\mu \cdot A^{a\mu} = 0 \qquad (\text{"axial gauge"})$$

are popular choices.

Equally well, we can impose the condition

$g(A^\alpha) = \omega(x)$, where $\omega(x)$ is some arbitrary function of x and

$$(A^\alpha)^a_\mu t^a = V(x) \left[A^a_\mu t^a + \frac{i}{g} \partial_\mu \right] V^\dagger(x)$$

$$V(x) = \exp(i \alpha^a t^a)$$

This is now the analogue of $y_\theta = 0$.

To preserve gauge invariance, we integrate over all $\alpha^a(x)$, in the same way we integrated over all angles

$\prod_x \delta(g(A(x)) - w(x))$

$$\int \mathcal{D}\alpha \, \delta(g(A^\alpha) - w) \det\left(\frac{\delta g(A^\alpha)}{\delta \alpha}\right) = 1$$

[= 1 if the solution is unique. It is not: there are so called "Gribov-copies", however they do not contribute in PT.

$$Z = \int \mathcal{D}\alpha \int \mathcal{D}A_\mu \, \delta(g(A^\alpha) - w) \det\left(\frac{\delta g(A^\alpha)}{\delta \alpha}\right) \exp\left[i S[A] \right]$$

Now change variables

$$t^a A^a_\mu \to t^a A^{a\prime}_\mu = t^a (A^\alpha)^a_\mu = V \left[A^a_\mu t^a + \frac{i}{g} \partial_\mu \right] V$$

$$Z = \left[\int \mathcal{D}\alpha\right] \int \mathcal{D}A'_\mu \; \delta(g(A') - \omega) \overbrace{\det\left(\frac{\delta g(A')}{\delta\alpha}\right)}^{\alpha\text{-indep!}} \exp\left[i S[A'] \right]$$

So we have factored out the integral over the gauge group. We still want to bring the path integral into managable form.

1.) Note that Z is independent of ω, so let's integrate over it with Gaussian weight

$$Z = \frac{1}{N} \int \mathcal{D}\omega \, e^{-i \int d^4x \frac{\omega^2}{2\xi}} Z$$

e.g. $(\partial_\mu A^\mu)^2$

$\downarrow$

$$= \frac{1}{N} \left[\int \mathcal{D}\alpha\right] \int \mathcal{D}A_\mu \exp\left[i S[A] - i \int d^4x \frac{1}{2\xi} g(A')^2 \right]$$

2.) Let's represent $\det\left(\frac{\delta g(A)}{\delta\alpha}\right)$ as a functional integral.

$$\det\left(\frac{\delta g^a}{\delta\alpha^b}\right) = \int \mathcal{D}\bar{c}\,\mathcal{D}c\,\exp\Big[i\int d^4y\int d^4z \\ -\bar{c}^a(y)\left[\delta g^a(y)/\delta\alpha^b(z)\right]c^b(z)\Big]$$

The fields c & $\bar{c}$ are the Feynman-DeWitt-Faddeev-Popov ghosts. They are scalars under the Lorentz group, but anti-commuting!

Let's calculate the functional derivative $\delta g^a(A(y))/\delta\alpha^b(z)$ for Lorentz gauge

$$t^a g^a(A^\alpha) = \partial_\mu A^{\alpha a}_\mu t^a = \partial^\mu V(x)\left[A^a_\mu t^a + \frac{i}{g}\partial_\mu\right]V^\dagger(x)$$

$$= \partial^\mu\Big[A^a_\mu t^a + \frac{1}{g}\partial_\mu\alpha^a t^a + \underbrace{i\left[\alpha^a t^a, A^b_\mu t^b\right]}_{-f^{bca}\alpha^b A^c_\mu t^a}\Big]$$

$$g^a(A^\alpha) = A^a_\mu + \frac{1}{g}\partial_\mu D^\mu\alpha^a$$

$\partial_\mu + i t^a A^a$

$$\frac{\delta g^a(A^\alpha(y))}{\delta\alpha^b(z)} = \frac{1}{g}\partial^\mu \underbrace{D_\mu}_{\delta^{ab}\partial_\mu - g f^{abc}A^c_\mu}\delta(y-z)$$

To summarize

have absorbed $\frac{1}{g}$ into c-field

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2}\xi(\partial^\mu A^a_\mu)^2 + \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b + \bar{\Psi}(i\not{D} - m)$$

$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu$

Note for QED $D^{ab}_\mu \Rightarrow \partial_\mu$: the ghosts can be integrated out since they don't interact.

We know that the theory is gauge invariant, but it is no longer manifest. However, even the gauge fixed Lagrangian has a symmetry, the so called BRST (Becchi, Rouet, Stora & Tyutin) symmetry (see e.g. Peskin) which can be used to derive Ward identities.

The ghost contributions are unphysical and cancel the equally unphysical contributions from timelike and longitudinal polarizations of the gluon.

In QED, the unphysical polarizations do not appear, as long as all external photons are transversly polarized. However in QCD they appear in loop diagrams. The ghost loops cancel their contribution. (Note that ghost loops involve a factor -1 because the fields anticommute.)