2.3 Lie groups, SU(N)

Lie groups represent continuous symmetries, e.g. rotations. To build gauge theories, we are interested in groups with finite-dim unitary representations, so called compact Lie groups.

We are interested in groups which can be generated from repeated action of an infinitesimal group elements

"generators"

$$g(\alpha) = 1 + i\alpha^a t^a + O(\alpha^2)$$

The structure of the group is determined by the commutation relations

$$[t^a, t^b] = i f^{abc} t^c$$

"structure constants"

Note Baker - Campbell Hausdorff

$$e^{i\alpha\cdot t} e^{i\beta\cdot t} = e^{i(\alpha+\beta)\cdot t - \frac{1}{2}[\alpha\cdot t, \beta\cdot t] + \dots}$$

Terminology for Lie groups

Simple: Cannot split generators in two commuting sets

Semi-simple: Does not contain generators which commutes with all others, ie. no $U(1)$ factors.

Jacobi identity

$$[t^a,[t^b,t^c]]+[t^b,[t^c,t^a]]+[t^c,[t^a,t^b]] = 0$$

implies

$$[[T^a, i f^{bcd} T^d]] = i f^{ade} f^{bcd} T^e + \dots$$

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$$

There exist only a small number of such groups (Killing & Cartan)

1.) <u>SU(N)</u> $\vec{\xi} \to U\vec{\xi}$ (ξ complex)

Complex n-dim unitary matrices $U^+U = 1$ with $\det(U) = 1$

$N^2 - 1$ generators, which are Hermitian traceless matrices $T^+ = T$ $\mathrm{tr}(T) = 0$

2.) Orthogonal transformations $UU^{+} = 1$ of n-dim vectors. Rotation group.

$N(N-1)/_2$ generators

3.) Symplectic transformations of N-dim. vectors $Sp(N)$. These trafo's leave the product

$$\vec{y}^T E \vec{x} \quad \text{invariant} \quad E = \begin{pmatrix} 0 & \mathbb{1}_{\frac{N}{2}\times\frac{N}{2}} \\ -\mathbb{1}_{\frac{N}{2}\times\frac{N}{2}} & 0 \end{pmatrix}$$

It has $N(N+1)/_2$ generators

In addition, there are five exceptional Lie algebras denoted by

G_2, F_4, E_6, E_7, E_8

A d-dim representation R is a set of d x d matrices which satisfy the commutation relations.

Choose t_R^a such that

$$\mathrm{tr}\left[t_R^a t_R^b \right] = T_R \ \delta^{ab}$$

For each representation, there is a complex conjugate rep.

$$\phi \longrightarrow (1 + i\alpha^a t^a_R)\phi$$

$$\phi^* \longrightarrow (1 - i\alpha^a (t^a_R)^*)\phi$$

$\rightsquigarrow$ $t^a_{\bar{R}} = -(t^a_R)^* = -(t_R)^T$.

The structure constants f_{abc} form a representation, the so-called adjoint repr.

$$(t^c_A)_{ab} = -i f_{abc}$$

The commutation relations for t^c_A are the Jacobi identity.

The simplest representation for $SU(N)$ is on the space of N-dim vectors and is called the fundamental rep.

We choose t^a, such that

$$(*) \quad \mathrm{tr}\left[t_F^a t_F^b \right] = T_F \, \delta^{ab} = \frac{1}{2} \delta^{ab}$$

The Casimir operator $T_R^2 = \sum_a t_R^a t_R^a$ commutes with all group elements. For an irreducible representation, it follows (Schur's Lemma)

$$T_R^2 = C_R \cdot \mathbb{1} .$$

[Irreducible : block diagonalize representation. If there is only 1 block, the rep. is irred.

Let's calculate C_F for $SU(N)$

$$\mathrm{tr}\left[t_F^a t_F^a \right] = \frac{1}{2} \sum_{a,b} \delta^{ab} = \frac{1}{2}(N^2 - 1)$$

$$= \mathrm{tr}\left[C_F \, \mathbb{1} \right] = C_F \, N$$

$$\rightsquigarrow \quad C_F = \frac{N^2 - 1}{2N}$$

Without proof $C_A = N$.

An explicit basis of generators for $SU(3)$

are the Gell-Mann matrices

$$t^1 = \frac{\lambda^1}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

... (see any book)

$$t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}.$$

However, in practice one avoids the explicit rep. and tries to express everything in terms of Casimir invariants, or uses the Fierz relation of $SU(N)$

$$t^a_{ij}\, t^a_{kl} = \frac{1}{2}\left(\delta_{il}\, \delta_{kj} - \frac{1}{N}\, \delta_{ij}\, \delta_{kl} \right)$$