2.2. The Yang-Mills Lagrangian

'54 Yang and Mills proposed to generalize invariance under phase rotation to an arbitrary continuous group (Lie group).

The simplest example is the rotation group $SU(2)$ ($\cong O(3)$). Consider $\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ and demand invariance under

$$\psi(x) \longrightarrow \exp\left(i\,\alpha^i(x)\frac{\sigma^i}{2}\right)\psi(x) = V(x)\,\psi(x)$$

The σ^i are the usual Pauli matrices:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] \underline{\underline{=}}\; i\varepsilon^{ijk}\frac{\sigma^k}{2} \quad ; \quad \sigma^i = (\sigma^i)^+ \quad \mathrm{tr}(\sigma^i) = 0$$

<u>Note</u>: V is unitary $VV^+ = 1$ and $\mathrm{Det}\, V = 1$

When discussing spin, the Pauli matrices describe rotations in real space. In contrast, we are considering rotations in field space, i.e. rotations of the fermions ψ_1 & ψ_2 into each other.

We now proceed in exactly the same way as last time. We introduce a link field

$$U(y,x) \Rightarrow V(y)\, U(y,x)\, V^{\dagger}(x)$$

$$\left[\, U(x,x) = 1 \,, \quad U^{\dagger}U = 1 \,, \quad U(x,y) = U^{\dagger}(y,x) \,\right]$$

and define the covariant derivative

$$n^{\mu} D_{\mu} \psi(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi(x+n\varepsilon) - U(x+n\varepsilon, x)\, \psi(x) \right]$$

and expand the link field:

$$D_{\mu} = \partial_{\mu} - i g A_{\mu}^{i} \frac{\sigma^{i}}{2}$$

Instead of a single photon A_μ, we get <u>three</u> gauge fields A^i_μ. (In the weak interaction, they correspond to $W^\pm, Z$.)

Let's derive the transformation law for A^i_μ.

$$U(x+n\varepsilon, x) = 1 + i\varepsilon g n^\mu A^i_\mu \frac{\sigma^i}{2}$$

$$\longrightarrow V(x+\varepsilon n)\left[1 + ig\varepsilon n^\mu A^i_\mu \frac{\sigma^i}{2}\right] V^+(x)$$

$$V(x+\varepsilon n)\, V^+(x) = \left[(1+\varepsilon n^\mu \partial_\mu)\, V(x)\right] V^+(x)$$

$$= 1 - \varepsilon n^\mu V(x)\, \partial_\mu V^+(x)$$

So $$A^i_\mu \frac{\sigma^i}{2} \longrightarrow V(x)\left[A^\mu_i \frac{\sigma^i}{2} + \frac{i}{g}\partial^\mu\right] V^+(x)$$

Complicated! Expand $V(x) = 1 + i\alpha^i \frac{\sigma^i}{2} + O(\alpha^2)$.

$$A^i_\mu \frac{\sigma^i}{2} \longrightarrow A^\mu_i \frac{\sigma^i}{2} + \underline{\frac{1}{g}}\,\partial^\mu \alpha^i \frac{\sigma^i}{2} + i\underbrace{\left[\alpha^i \frac{\sigma^i}{2}, A^j_i \frac{\sigma^j}{2}\right]}_{\text{new!}}$$

Exercise: check that $D_\mu \psi \to V D_\mu \psi$.

To obtain the kinetic term for the gauge field, we again consider $[D_\mu, D_\nu]\psi(x)$.

As in the abelian case, $[D_\mu, D_\nu]\psi$ does not contain a derivative on ψ.

$$[D_\mu, D_\nu] = i g F^i_{\mu\nu} \frac{\sigma^i}{2}$$

$$F^i_{\mu\nu} \frac{\sigma^i}{2} = \partial_\mu A^i_\nu \frac{\sigma^i}{2} - \partial_\nu A^i_\mu \frac{\sigma^i}{2} - i g \left[A^i_\mu \frac{\sigma^i}{2}, A^j_\nu \frac{\sigma^j}{2} \right]$$

Now use: $\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2} \right] = i \varepsilon^{ijk} \frac{\sigma^k}{2}$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g \varepsilon^{ijk} A^j_\mu A^k_\nu$$

The transformation law $[D^\mu, D^\nu] \to V [D^\mu, D^\nu] V^+$ implies that $F^i_{\mu\nu}$ is not gauge invariant.

However $\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\left[\left(F^i_{\mu\nu}\frac{\sigma^i}{2}\right)^2\right] = -\frac{1}{4}(F^i_{\mu\nu})^2$

is gauge inv.

sum over i

The most general renormalizable $\mathcal{L}$ is

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - m\bar{\psi}\psi - \frac{1}{4}(F^i_{\mu\nu})^2 \underbrace{- i c\, \varepsilon^{\mu\nu\rho\sigma} F^i_{\mu\nu} F^i_{\rho\sigma}}_{\text{violates } P,\, T}$$

Our discussion easily generalizes to an arbitrary continuous symmetry group.

Yang & Mills wanted to explain the strong interaction with an SU(2) gauge field acting on $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$. However the gauge fields are massless, so why were they not observed? (Also, it was not clear at the time how to quantize YM theory.)