2. Non-abelian gauge theory

QCD is an example of a non-abelian gauge theory. (The electroweak theory is another example!) These theories are generalizations of electrodynamics which is an abelian gauge theory.

2.1. Gauge invariance

Consider complex field $\psi(x)$ and perform phase redefinitions

$$\psi(x) \to e^{i\alpha(x)}\,\psi(x) \qquad \text{"gauge transformation"}$$

We now want to construct a theory which is invariant under this transformation. $\psi^{+}(x)\cdot\psi(x)$ is invariant.

However, derivatives no longer make sense

$$n^\mu \partial_\mu \psi(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\psi(x+n\varepsilon) - \psi(x)\right]$$

unit vector

Different transformation law.

In order to be able to compare fields at different points we need to introduce a quantity which compensates phase difference at neighbouring points, e.g. "Link field"

$$U(y,x) \Rightarrow e^{i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)}$$

$$\left[\, U(x,x) = 1 \qquad U(x,y) = U^\dagger(y,x) \qquad |U| = 1 \,\right]$$

Then $\quad n^\mu D_\mu \psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\psi(x+\varepsilon n) - U(x+n\varepsilon, x)\psi(x)\right]$

Expand $\quad U(x+n\varepsilon, x) = 1 - \varepsilon\, i e\, n^\mu A_\mu(x) + O(\varepsilon^2)$.

The field A_μ is called a <u>connection</u> (or gauge field)

Transformation: $\quad A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$

Check:

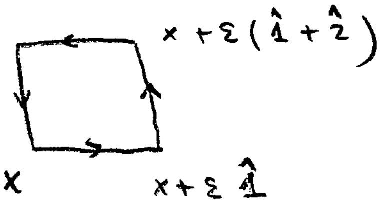
$$D_\mu \psi \Rightarrow \{ \partial_\mu + ie (A_\mu - \tfrac{1}{e} \partial_\mu \alpha) \} e^{i\alpha} \psi$$

$$= e^{i\alpha} \{ \partial_\mu + ie A_\mu \} \psi = e^{i\alpha} D_\mu \psi$$

We can now construct a gauge invariant Lagrangian for ψ which includes derivative terms.

In terms of $U(y,x)$ we could consider closed curves

e.g.



To get an invariant using differential quantities consider

$$[D_\mu, D_\nu] \psi(x) \rightarrow e^{i\alpha} [D_\mu, D_\nu] \psi$$

$$[D_\mu, D_\nu]\psi = \Big([\partial_\mu, \partial_\nu] + ie \big(\overset{(\partial_\mu A_\nu)}{[\partial_\mu, A_\nu]} - [\partial_\nu, A_\mu] \big) - e^2 \underbrace{[A_\mu, A_\nu]}_{0} \Big) \psi$$

$$= ie (\partial_\mu A_\nu - \partial_\nu A_\mu) \psi$$

So $[D_\mu, D_\nu] = ie F_{\mu\nu}$

We can now write down the most general renormalizable Lagrangian, i.e. all operators up to dimension 4.

$[\psi] = 3/2 \quad , \quad [D_\mu] = 1$

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - m \bar{\psi}\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$- i c \underbrace{\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}}_{\text{violates } P, T}$$