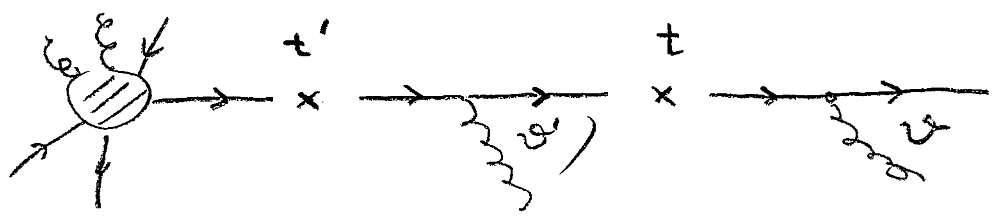
10.2. Parton shower

In the last section we have derived a relation between the cross section for n+1 partons in terms of the n-parton cross section in the collinear limit:

$$d\sigma_{n+1} = d\sigma_n \frac{\alpha}{2\pi} \frac{dt}{t} dz \, P_{i\to jk}(z) \frac{d\phi}{2\pi} \quad ; \, t = p^2$$

Here, $P_{i\to jk}$ is the splitting function for the $i\to jk$ branching ($q\to qg$, $g\to q\bar{q}$, $g\to gg$, $\bar{q}\to\bar{q}g$).
The splitting depends weakly on ϕ and also on the polarizations. For simplicity we'll suppress this dependence.

We can obtain the cross section for multiple emissions by iterating the above relation:



+ " gluon branchings "

$$d\sigma_{n+2} = \sum_{x,y} d\sigma_n \frac{\alpha(t')}{2\pi} \frac{dt'}{t'} P_x(z')\, dz' \frac{d\phi'}{2\pi}$$

$$\times \frac{\alpha(t)}{2\pi} \frac{dt}{t} P_y(z)\, dz \frac{d\phi'}{2\pi} \quad \Theta(t'-t)$$

The sum over x,y indicates that one should sum over all splittings which lead to the same final state.

Unfortunately, the results are not yet in useable form. If we would want to calculate a jet cross section, simply adding our tree-level results

$$\sigma_n^{tree} + \sigma_{n+1}^{tree} + \sigma_{n+2}^{tree} + \dots$$

gives the wrong result. At the same order in α_s as σ_{n+1}^{tree} also σ_n^{1-loop} will contribute, etc.

However, using unitarity, one can obtain also the virtual corrections at the same level of accuracy.

The amplitudes squared $|M|^2$ are probabilities for certain processes to happen. The splitting functions can therefore be viewed as probabilities for an additional emission

$$dP_{emis}(t+dt, t) = \frac{dt}{t} \frac{\alpha(t)}{2\pi} \int dz \, P_{i\to jk}(z)$$

So the probability not to have an emission is

$$dP_{no\ emis}(t+dt, t) = 1 - \sum_{jk} dP_{emis}$$

$$= 1 - \sum_{j,k} \frac{dt}{t} \frac{\alpha(t)}{2\pi} \int dz \, \hat{P}_{i\to jk}(z)$$

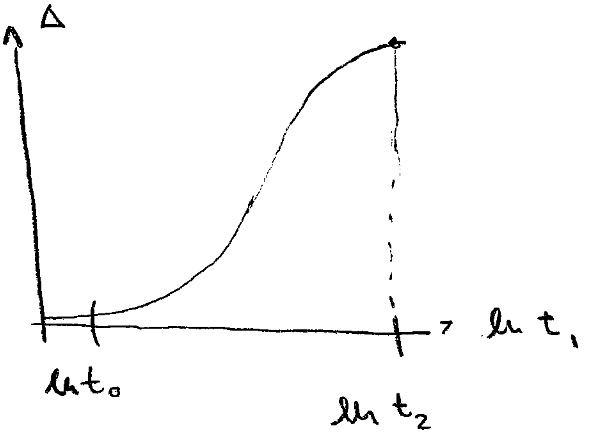
If we divide an interval $[t_1, t_2]$ in N small intervals $dt = (t_2 - t_1)/N$, then

$$P_{no\ emis}(t_2, t_1) = \lim_{N\to\infty} \prod_{n=1}^{N} \left[1 - \sum_{jk} \frac{dt}{t_n} \frac{\alpha(t_n)}{2\pi} \int dz \, \hat{P}(z) \right]^N$$

$$\to \exp\left\{ - \int_{t_1}^{t_2} \frac{dt}{t} \frac{\alpha(t)}{2\pi} \int dz \sum_{ke} \hat{P}_{i\to k,e}(z) \right\} \equiv \Delta_i(t_2, t_1)$$

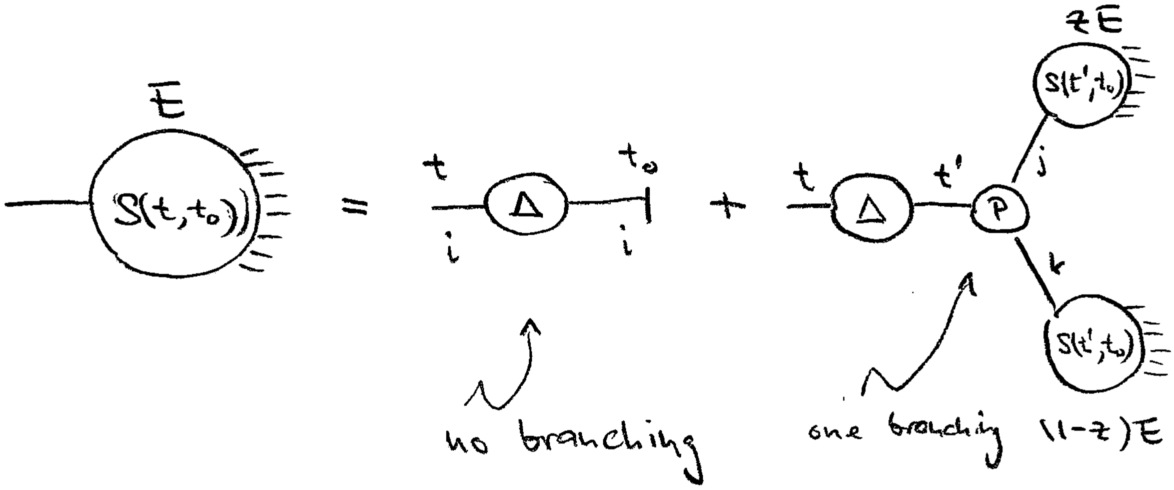
$\Delta_i(t_2, t_1)$ is called the Sudakov form factor.

Very roughly $\Delta(t_2, t_1) \sim \exp\left[-\frac{\alpha}{2\pi} \ln^2\left(\frac{t_2}{t_1}\right)\right]$



The probability for a parton with virtuality t_2 to not shower down to a small scale t_0 is very small

Now we can formulate the parton shower S



This form is now implemented into a computer code which will generate emissions according to probability, and terminate once it reaches t_0.

Let us discuss the computer implementation in some detail.

If we start from a given value of t, the probability to have an emission at t' is

$$dP = \Delta_i(t,t') \frac{\alpha(t')}{2\pi} \frac{dt'}{t'} \int \sum_{(jk)} P_{i\to jk}(t)\,dz \frac{d\varphi}{2\pi}$$

$$= d\Delta(t,t') \left[= \frac{d\Delta(t,t')}{dt'} dt' \right]$$

We would like our computer code to generate random values t' for the next branching distributed as dP, starting with a uniform random variable r in $[0,1]$

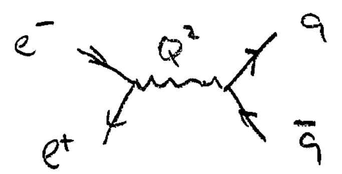
$$dP = f(t')\,dt' = 1\,dR \quad \text{, with} \quad f(t')dt' = dF(t')$$

$$\int_{t'}^{t} dt''\, f(t'') = \Delta(t,t') = \int_0^r 1\,dR = r$$

→ To get the value of t' generate random value $r \in [0,1]$ and solve $\Delta(t,t') = r$ for t' numerically.

Now we are in a position to formulate our shower algorithm:

1.) For each final-state colored parton, generate a shower with $t = Q^2$ (where Q^2 is a typical high scale of the process).



2.) For each shower: generate random number $0 < r < 1$.

Solve $r = \Delta_i(t, t')$ for t'

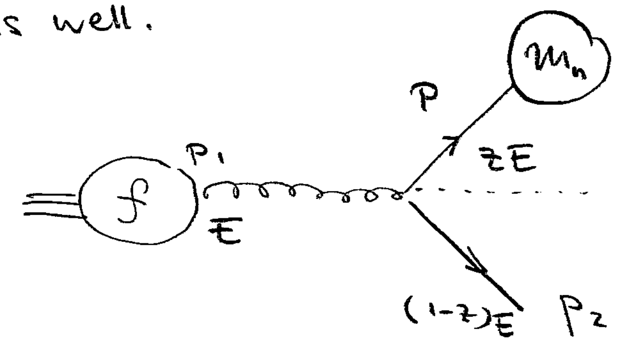
2a.) If $t' < t_0$ (cut-off) stop the shower

2b.) If $t' \geq t_0$ generate z, jk with probability $P_{i,jk}(z)$ and $0 < \varphi < 2\pi$ uniformly.

Assign $E_j = z E_i$; $E_k = (1-z) E_i$

3.) Restart shower for j and k, setting $t = t'$.

At a hadron collider, we also have incoming color-charged partons and would like to shower them as well.



$t = p^2$

Despite the changed kinematics, the basic formula

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha}{2\pi} P_{i\to jk}$$

is exactly the same as for the outgoing legs.

Note that additional emissions increase the value of t for radiation off incoming legs. When implementing initial state showering, it is most efficient to evolve also in this case from large t to smaller t. The probability for this backward evolution is obtained by solving

$$\frac{f_i(\xi, \mu^2 = t')\,\Delta(t,t')}{f_i(\xi, \mu^2 = t)} = r$$

where the parton i carries the momentum fraction $\xi = \frac{E_i}{E_h}$. For an explanation, see Sjöstrand PLB 157:321, 1985.

Our shower evolves from large virtuality $t = P^2$ to lower values, but there are other possible choices such as the opening angle $t = E^2\Theta^2$, or the transverse momentum $t = p_T^2$. They are related as follows

$$P^2 = z(1-z)E^2\, 2(1-\cos\Theta) \cong z(1-z)\Theta^2 E^2$$

$$p_T^2 = E_1^2 \sin^2\Theta_1 \cong z^2\Theta_1^2 E^2 = z^2(1-z)^2\,\Theta^2 E^2$$

$z^2 < 1$

Note that $z(1-z) > \frac{P^2}{E^2}$ so that $z \in [\frac{t}{E^2}, 1-\frac{t}{E^2}]$ for a virtuality ordered shower and $z \in [\frac{\sqrt{t}}{E}, 1-\frac{\sqrt{t}}{E}]$ for a p_T^2-ordered shower. In the following, we'll see that only angular ordering gives correct results also for soft emissions. Nevertheless, Pythia, the most popular MC program, uses p_T-ordering (and used to use virtuality ordering).